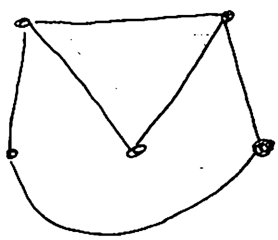
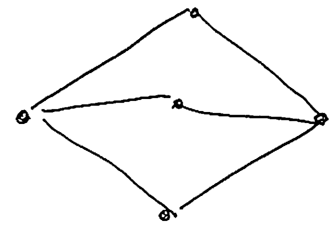
Solutions HW 3

1) Consider the seq: 33222

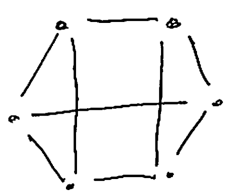


and

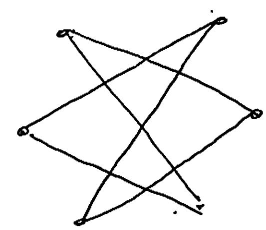


The first has its two deg 3 vertices adj, the 2nd does not.

2) Consider graph complements:

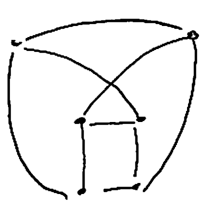


$\Longrightarrow$

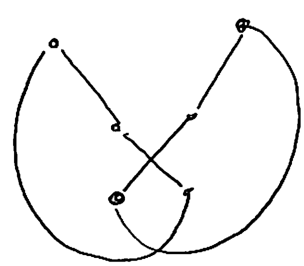


$= C_6$

But

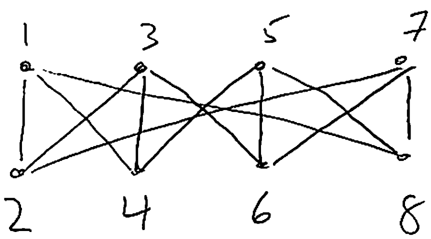


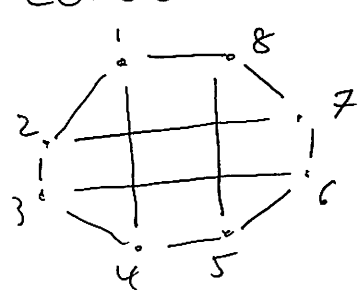
$\Longrightarrow$



$= C_3 + C_3$

3) The first two are $\cong$ consider





The third has a cycle of odd length, so is not bipartite like the first two.

4) Observe that if we have n vertices then we have $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of vertices.

As each labeled graph is determined by either drawing an edge between each pair or not drawing an edge we see that the # of such options (& hence graphs) is

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{\text{\# pairs of vertices}} = 2^{\binom{n}{2}}.$$

5) a- Let $u \in V(G)$. Consider $N_G(u)$ + $N_H(\varphi(u))$.

If we can show

$$\varphi : N_G(u) \to N_H(\varphi(u)) \qquad (*)$$

bijectively then

$$\deg_G u = |N_G(u)| = |N_H(\varphi(u))| = \deg_H \varphi(u).$$

To show this we first show that if $v \in N_G(u)$ then $\varphi(v) \in N_H(\varphi(u))$.

If $v \in N_G(u)$, then $v \sim u$. By defn of our graph ish this means $\varphi(v) \sim \varphi(u)$ so $\varphi(v) \in N_H(\varphi(u))$ as needed.

Next we need to show this mapping $(*)$ is onto.

Let $y \in N_H(\varphi(u))$. As $\varphi : V(G) \to V(H)$ is onto there exists $x \in V(G)$ s.t. $\varphi(x) = y \sim \varphi(u)$. This means $x \sim u$ so $x \in N_G(u)$. Thus $(*)$ is a bijection. (we are given it is 1-1.)

b) Let $x, y \in V(G-u)$, now

$x \sim y$ in $G-u$ iff $x \sim y$ in G

iff $\varphi(x) \sim \varphi(y)$ in H iff $\varphi(x) \sim \varphi(y)$ in

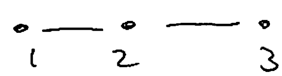
$G - \varphi(u)$.

c) We see that (from a) that all the vertices of deg d in G must map to all those of deg d in H. The same argument in b) then says we can delete these and the resulting graph is ish.

Connected Graphs

1) a- False, consider the walk

$(1\ 2\ 3\ 2\ 1)$

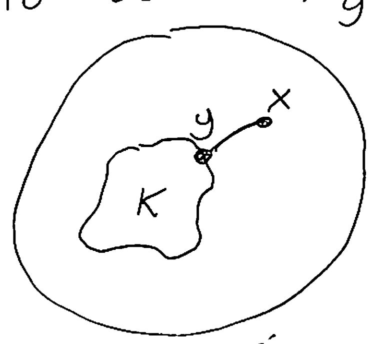
on 

b- True. If u and v are not cnt'd then

G: 

G has at least 2 components K_1 + K_2. But then K_1 is a graph w/ only 1 odd vertex $\Rightarrow\Leftarrow$.

2) Assume not for a contradiction. Let K be a comp. of $G-u$ ∍. u is not adj to any vertex in K. As G is cntd we must have some vertx, x that is not in K adj to a vertex y in K:

G: 

If $x \neq u$, then in $G-u$, $x + y$ are still adj. Thus they are in the same component of $G-u$. As $y \in V(K)$ then $x \in V(x)$ $\Rightarrow\Leftarrow$.

So $x=u$ as needed.

3) Let G be disconnected. Fix $x, y \in V(G)$.

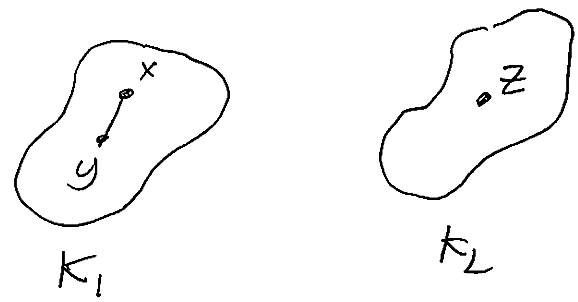
Case 1: $x \not\sim_G y$

Then in $\overline{G}$, $x \sim y$ so $\exists$ an x-y path in $\overline{G}$.

Case 2: $x \sim_G y$

As G is disconnected it must have at least 2 components and as $x \sim y$ then x & y are in the same component. So

G:

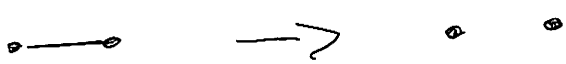


where z is a vertex in another comp. As $x \not\sim_G z$ and $y \not\sim_G z$ then in $\overline{G}$ we have

$$x \sim_{\overline{G}} z \sim_{\overline{G}} y$$

thus in $\overline{G}$ $\exists$ an x-y path, namely $(x\ z\ y)$.

b) The converse is false:



$K_2 \longrightarrow \overline{K_2} = K_1 + K_1$

4) G consists of two components, A & B.

where $A = \{ w \in G \mid \text{sum of all the digits in } w \text{ is even} \}$

$B = \{ w \mid \text{sum of all digits is odd} \}$.

5) Assume PG did have a cycle of length 4:

$$\begin{array}{ccc} A & - & B \\ | & & | \\ C & - & D \end{array}$$

where A, B, C, D are distinct 2-element subsets of D, w/ $A \cap C = A \cap B = C \cap D = B \cap D = \emptyset$.

So wlog $A = \{12\}$, then $B = \{34\}$ $C = \{45\}$.

Now D must be disjoint from B & C. The only way this can happen is for $D = \{12\} = A$. But we said all our sets were distinct. Thus no 4-cycle can exist.