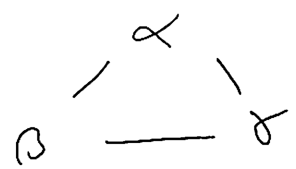
HW2 - Solutions

1) <u>Hypercube</u>

Recall that the vertex set of Q_n is the set of binary words of length n. Two words are adjacent iff they differ in <u>exactly</u> 1 position. Now assume for a contradiction that $K_3 \subseteq Q_n$. This means we have 3 words α, β & γ $\ni$



So α differs from β in exactly 1 position (say i)
and β " " γ " " " (say k).
As $\beta \neq \gamma$ we must have $i \neq k$. Now wlog assume α looks like

$$\alpha = \_\_\_\_ \underset{i}{0} \_\_\_\_ \underset{k}{0} \_\_\_\_$$

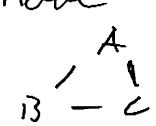
then $$\beta = \_\_\_\_ \underset{i}{1} \_\_\_\_ 0 \_\_\_\_$$

$$\gamma = \_\_\_\_ 0 \_\_\_\_ \underset{k}{1} \_\_\_\_$$

So β & γ differ in two positions! Thus $K_3 \not\subseteq Q_n$.

<u>Peterson Graph</u>

Again assume for a contradiction that $K_3 \subseteq PG$. So we must have 2-element subsets of $\{1 \ldots 5\}$ A, B, C $\ni$.



or $A \cap B = B \cap C = C \cap A = \emptyset$.

Note $A \cup B \cup C \subseteq \{1 \ldots 5\}$ so $|A \cup B \cup C| \leq 5$

But as these sets are mutually disjoint:

$$|A \cup B \cup C| = |A| + |B| + |C| = 6.$$

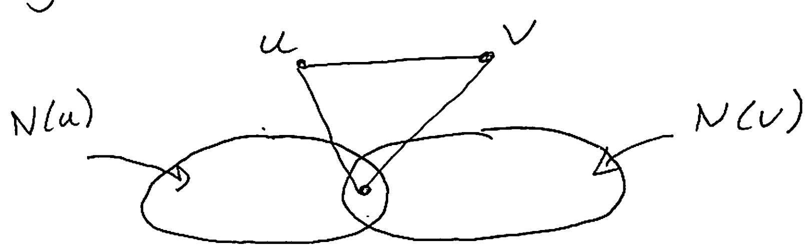
This contradiction implies $K_3 \notin PG$

2) a- The order of $G_{n,k}$ is just the # of k-element subsets of the set $\{1, 2, \dots, n\}$. This is $\binom{n}{k}$.

b- Yes, $G_{n,k}$ is regular. If A is any k-element subset of $\{1, \dots, n\}$ then all of its n'bors are just the k-element subsets that are disjoint from A. If $S = \{1, \dots, n\} \setminus A$, then

this is just the # of k-element subsets of S. This # is $\binom{n-k}{k}$, since $|S| = n-k$.

3) we are given that $uv \in E(G)$. So we have



So the # of triangles is exactly $|N(v) \cap N(u)|$.

If G has order n then

$$n \geq |N(v) \cup N(u)| = |N(v)| + |N(u)| - |N(v) \cap N(u)|$$

$$\Rightarrow \quad |N(v) \cap N(u)| \geq |N(v)| + |N(u)| - n$$
$$= \deg v + \deg u - n.$$

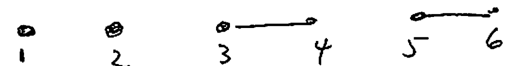
4) a- Not graphical.
↳ This has an odd # of odd degrees.

b- Here we use Havel-Hakimi (thm 2.3)

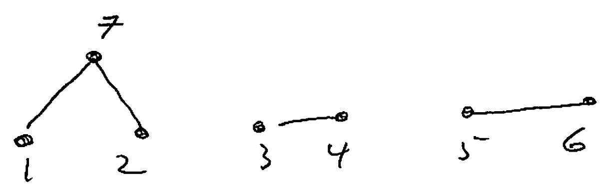
6 3 3 3 3 2 2 1 1
2 2 2 2 1 1 1 1 .
1 1 2 1 1 1 1 } reorder
. 2 1 1 1 1 1 1
0 0 1 1 1 1

As 001111 is graphical:  then the orig. seq is graphical too.

To construct a graph start w/

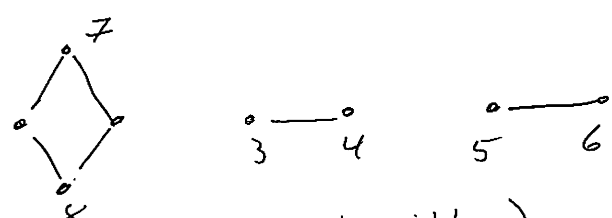


Now add a new vertx, 7, and connect it as follows:



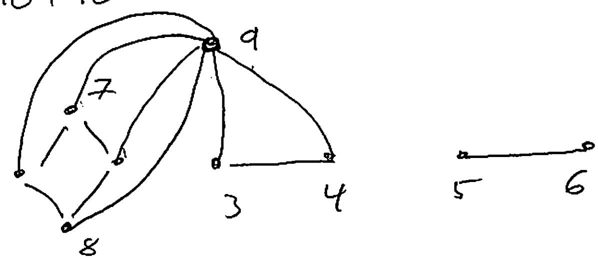
(this corresponds to the last step of the algorithm.)

Now connect a new vertex to the 1 + 2 again:



(this is the middle step of the algorithm)

Lastly connect another new vertex deS



(this represents the first step in the algorithm.)

5) To solve this problem we will use both induction & the Havel-Hakimi thm.

For the base case, let $n = 1$ so $a = 0$. This is certainly graphical, consider K_1.

Now (induction step) assume that any such seq of length $n \geq 1$ is graphical. Now consider such a seq

$$\underbrace{a \cdots a}_{k+1} \underbrace{(a-1) \cdots (a-1)}_{\ell} \qquad (1)$$

of length $1+k+\ell = n+1$. We are given that

① $n+1 > a$ and ② Its sum $a(k+1) + (a-1)\ell = s$ is even.

Next we apply the Havel-Hakimi Lemma. We have two situations depending on how a & $(k+1)$ compare.

Case: $k \leq a$

If we apply Havel-Hakimi we get

$$\underbrace{(a-1) \cdots (a-1)}_{k} \underbrace{(a-2) \cdots (a-2)}_{a-k} \underbrace{(a-1) \cdots (a-1)}_{\ell-a+k} \qquad (2)$$

This seq has length n, its sum is $s - 2a$ (why?) which is even and $n > a-1$. By induction (2) is graphical. Thus by Havel-Hakimi (1) is graphical.

Case: $k > a$

Again we apply Havel-Hakimi to (1). We get:

$$\underbrace{(a-1) \cdots (a-1)}_{a} \underbrace{a \cdots a}_{k-a} \underbrace{(a-1) \cdots (a-1)}_{\ell}. \qquad (3)$$

This seq has length n and its sum is even, just as before. By induction it ~~has~~ is graphical & hence by Havel-Hakimi (1) is graphical as well.

6) Let $S = \{d_1 \dots d_m\}$. Recall that K_{n+1} has $n+1$ vertices of degree n. Now set

$$C = (d_1+1) \cdots (d_m+1)$$

Then $\underbrace{K_{d_i+1} + \cdots + K_{d_i+1}}_{\frac{C}{(d_i+1)}}$ has $\frac{C}{(d_i+1)} \cdot (d_i+1) = C$ vertices all of degree d_i. (Note: Here $G+H$ just means the graph consisting of G and H. So

$K_3 + K_3 =$ ,

for example.) Therefore the graph we want is

$$G = \Big(\underbrace{K_{d_1+1} + \cdots + K_{d_1+1}}_{\frac{C}{(d_1+1)}}\Big) + \Big(\underbrace{K_{d_2+1} + \cdots + K_{d_2+1}}_{\frac{C}{(d_2+1)}}\Big) + \cdots + \Big(\underbrace{K_{d_m+1} + \cdots + K_{d_m+1}}_{\frac{C}{(d_m+1)}}\Big)$$

since the degree seq for G is then

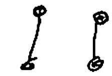
$$\underbrace{d_1, \dots, d_1}_{C}, \underbrace{d_2 \cdots d_2}_{C}, \dots, \underbrace{d_m, \dots, d_m}_{C}.$$

6) Let $S = \{d_1, \dots, d_m\}$. Note that K_{n+1} has $n+1$ vertices of degree n. So if we let G be the graph consisting of

$$\frac{(d_1+1)\cdots(d_m+1)}{(d_1+1)} \text{ copies of } K_{d_1+1}$$

7) we proceed by induction on k.

Base case: $k=1$

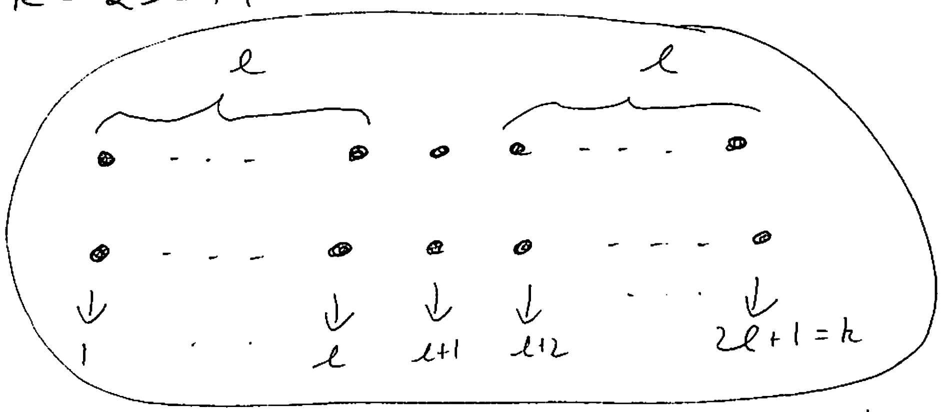
The seq $1\,1$ is clearly graphical: 

Now assume true for $k \geq 1$. By induction we know that $11\ldots kk$ is graphical, so there is a graph G w/ that deg seq.

Case 1: k is odd.

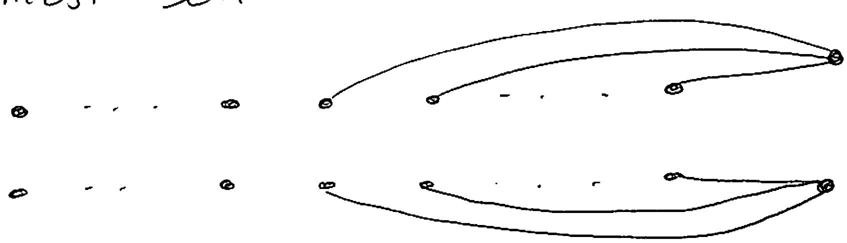
Let $k = 2\ell+1$ and draw G as

G



deg:

Now attach two new vertices and draw edges to the rightmost $\ell+1$ vertices in G:



These new vertices have deg $\ell+1$ and old vertices that are adjacent to these new vertices have deg $\ell+2, \ell+3, \ldots, k+1$.

Case 2: k is even

Let $k=2\ell$. Then the above argument "almost" works. instead:

$G \rightarrow$