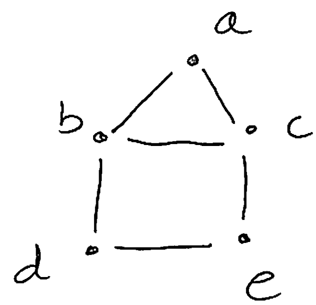
HW I: Solutions

1) a-

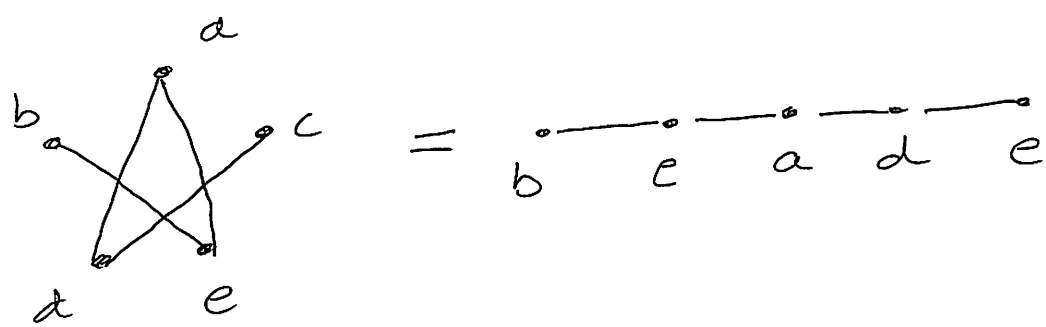


b- size = 6

order = 5

c- $\delta(G) = 2$, $\Delta(G) = 3$

d-



2) Observe that each vertex in K_n is adjacent to $n-1$ other vertices. It follows that each vertex has degree $n-1$. The handshake Lemma gives us that

$$n(n-1) = \sum_{v \in V(K_n)} \deg(v) = 2\underbrace{m}_{\#\text{ edges}}$$

$$\Rightarrow \quad m = \frac{n(n-1)}{2} = \binom{n}{2}.$$

3) By the handshake lemma we have

$$2 \cdot 10 = \underbrace{3 \cdot 2}_{\substack{3\text{ vertices}\\ \deg 2}} + \underbrace{2 \cdot 4}_{\substack{2\text{ vertices}\\ \deg 4}} + \underbrace{x \cdot 1}_{\substack{x\text{ vertices}\\ \deg 1}}$$

solving this yields:

$$20 = 6 + 8 + x$$
$$x = 6$$

4) We know from the Handshake Lemma that the sum of all the degrees in a graph must be an even #. Since the sum of an odd # of odd numbers is odd it follows that a graph must have an even # of vertices w/ odd degree.

5) Recall that the "if and only if" requires you to prove a fwd ($\Rightarrow$) and Reverse ($\Leftarrow$) direction.

Forward Direction ($\Rightarrow$)

If S is a clique in G, then S is indep in $\overline{G}$.

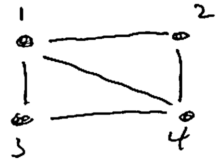
Pf: Assume S is a clique in G. This means that $\forall$ distinct vertices $u, v \in S$, $u \sim_G v$. By defn of $\overline{G}$, this implies that $u \not\sim_{\overline{G}} v$. We conclude that S is an indep set in $\overline{G}$.

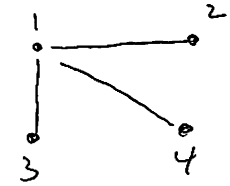
Reverse Direction ($\Leftarrow$)

If S is an indep set in $\overline{G}$, then S is a clique in G.

Pf: Assume S is an indep set in $\overline{G}$. This means that $\forall$ distinct vertices $u, v \in S$, $u \not\sim_{\overline{G}} v$. This implies that $u \sim_G v$. As this is true for all distinct vertices $u, v \in S$, we conclude that S is a clique in G.

6) There are lots of possible answers here. One is:

$G =$ 

$H =$ 

$\Rightarrow H \subseteq G$ but H is not an induced subgraph.

7) Assume, for a contradiction, that G is a graph of order 6 w/ neither a clique nor an indep set of order 6.

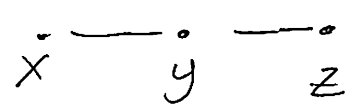
a) First consider three vertices
$$\{x, y, u\}.$$

As this set cannot form an indep set we must have at least one edge btwn them. without loss of generality (wlog) assume
$$x \sim y.$$

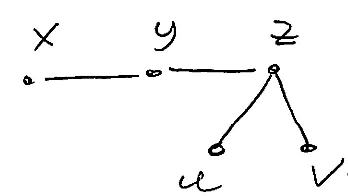
Now consider the vertices
$$\{u, v, w, z\}.$$

As we are not allowed to have cliques of order 3 it follows that two of these vertices are not adj. say $w \not\sim z$.

Lastly, consider the set of vertices $\{y, w, z\}$. As this cannot be an indep set & $w \not\sim z$ we must have either $y \sim w$ or $y \sim z$. Again wlog assume $y \sim z$. Thus we have:



b) From the preceding paragraph we know



As $\{y, u, v\}$ is not indep by assumption, then we must have at least one edge btwn them. Either way this forms a clique of order 3.