Math 428
Graph Theory
Homework Set #1

Basic Definitions

1. Consider the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{ab, ac, bc, bd, de, ec\}$.

   a) Draw the graph defined by G.

   b) What is the size and order of G?

   c) It is standard notation in graph theory to let $\delta(G)$ be the minimum degree and $\Delta(G)$ be the maximum degree of any graph G. What is $\delta(G)$ and $\Delta(G)$ in this example?

   d) Draw the picture of $\overline{G}$.

2. Give a formula for the number of edges in K_n.

3. Let G be a graph of size 10. Assume it only has vertices of degree 1, 2 and 4. If we know it has 3 vertices of degree 2, and 2 vertices of degree 4, then how many vertices of degree 1 must G have?

4. Prove that in any graph G there must be an even number of vertices with odd degree.

5. Let S be a set of vertices in a graph G. Show that S is a clique in G if and only if S is an independent set in $\overline{G}$.

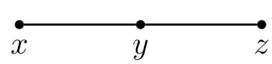
6. Let $H \subseteq G$. We say H is an *induced subgraph of* G if for every $u, v \in V(H)$, then uv is an edge of H if and only if uv is an edge of G.

   Find graphs H and G such that $H \subseteq G$ but is *not* an induced subgraph of G.

7. In class we discussed the Acquaintance Theorem which states the following: In any graph G on 6 vertices x, y, z, u, v, w, either there exists an independent set of order 3 or a clique of order 3. This problem will walk you through a proof of this (famous) theorem.

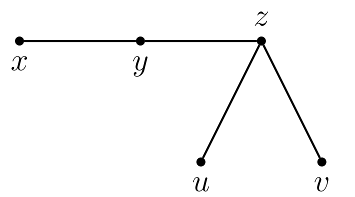
   First, assume for a contradiction that G has neither a clique of order 3 nor an independent set of order 3.

   a) Show there must exist 3 vertices x, y, z in G such that



.

From part a) we conclude that $x \nsim z$ since otherwise, x, y, z would be a clique of order 3, which we are assuming does not exist. Now consider the remaining three vertices u, v, w. Since u, x, z cannot be an independent set of order 3, and $x \nsim z$, we conclude that $u \sim x$ or $u \sim z$. The same argument shows that $v \sim x$ or $v \sim z$, and $w \sim x$ or $w \sim z$. It follows that two of the vertices u, v, w must be adjacent to either x or z. Without loss of generality (wlog) we may assume that



b) Conclude that there must exist a clique of order 3, which is our contradiction.