Modified Growth Diagrams, Permutation Pivots, and the BXW Map ϕ^*

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Overview

In their paper [1] on Wilf-equivalence for singleton classes, Backelin, Xin, and West introduce a transformation ϕ^*, defined by an iterative process and operating on (all) full rook placements on Ferrers boards. In [3], Bousquet-Mélou and Steingrímsson prove the analogue of the main result of [1] in the context of involutions, and in so doing they must prove that ϕ^* commutes with the operation of taking inverses. The proof of this commutation result is long and difficult, and Bousquet-Mélou and Steingrímsson ask if ϕ^* might be reformulated in such a way as to make this result obvious. In the present paper we provide such a reformulation of ϕ^*, by modifying the growth diagram algorithm of Fomin [4,5]. This also answers a question of Krattenthaler [6, problem 4], who notes that a bijection defined by the unmodified Fomin algorithm obviously commutes with inverses, and asks what the connection is between this bijection and ϕ^*.

1. Definition of ϕ^*

The map $\phi^* : S_n \to S_n(k \dots 1)$ is most clearly defined via an example. Fix $k = 3$ and $\pi = 45312$.

First identify the left-most $321-$pattern in σ and cycle this pattern forward:

$$645321 \longrightarrow \phi(\pi) = 435621$$

Now we repeat the process until it naturally stops:

$$435621 \longrightarrow \phi^2(\pi) = 325641$$

$$325641 \longrightarrow \phi^3(\pi) = 215643$$

$$215643 \longrightarrow \phi^4(\pi) = 214635$$

Finally we have our definition

$$\phi^*(645321) = \phi^4(\pi) = 214635$$

Next observe that $(645321)^{-1} = 654231$ and a similar calculation yields that

$$\phi^*((645321)^{-1}) = \phi^*(654231) = 215364 = (214635)^{-1} = \phi^*(645321)^{-1}$$

2. The Robinson-Schensted Correspondence

Recall that the Robinson-Schensted Correspondence sets up a bijection

$$S_n \longleftrightarrow Y_n^2$$

where Y_n^2 denotes the set of all pairs of standard Young tableaux that have the same shape. The next well-known theorem plays a central theme in our reformulation of ϕ^*.

Theorem: (Schensted)
Fix a permutation π and let (P, Q) be its image under this correspondence. If λ is the shape of P and Q then λ_1 is the length of a longest increasing subsequence in π.

3. Growth Diagrams

Our reformulation of ϕ^* will be accomplished by modifying Fomin's [4,5, see also 6]] construction (GDA) of the growth diagram of a rook placement P on a Ferrers board F.

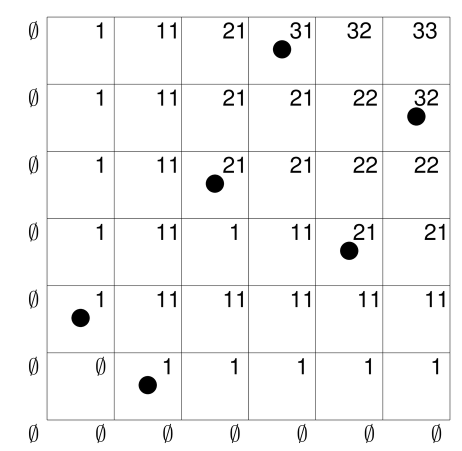
Fomin's construction assigns partitions to the corners of all the squares in F, using the markers of P. We start by assigning the empty partition $\emptyset$ to each corner on the left and bottom edges of F. We then assign partitions to the other corners inductively. Assuming that the northwest, southwest, and southeast corners of a square (i, j) have been assigned partitions $NW, SW,$ and SE, we assign to the northeast corner the partition NE determined by the following rules.

a. $(NW \neq SE)$ b. $(NW = SW = SE)$ c. $(NW = SW = SE)$ d. $(SW \neq NW = SE)$

Fact: The partition assigned to the northeast corner of (i, j) is the shape of the Robinson-Schensted tableaux for the partial permutation resulting from the restriction of P to the rectangle $R(i, j)$.

Example:

$\mathrm{GDA}(\phi^*(645321)) =$



4. Our Reformulation of ϕ^*

Our observation was to modify the growth diagram algorithm in the following way. We retains rules (a), (b), and (c) but replace rule (d) by the following variant.

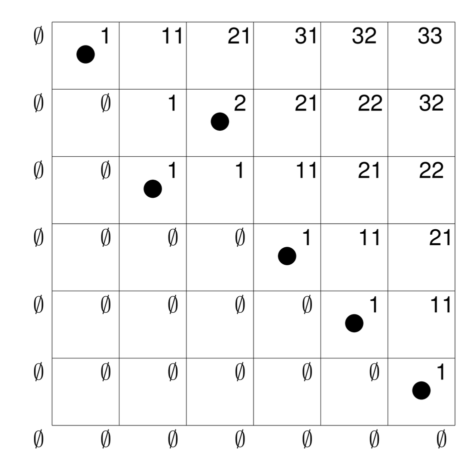
Rule d_k : If rule (d) produces a partition with k (nonzero) entries then delete the rightmost entry and increase the leftmost entry by 1.

We will refer to this modified algorithm as GDA_k.

Our motivation for the rule d_k comes from the theorem of Schensted mentioned earlier. Keeping the number of entries in a partition λ less than k prevents a decreasing subsequences of length k. The amazing fact is that this method prevents decreasing subsequences in the exact same manner as does ϕ^*!

To see an example of the reformulation of ϕ^* consider:

$\mathrm{GDA}_3(645321) =$



Note that the partitions across the top and right hand side are identical with the partitions computed across the top and right hand side for $\phi^*(645321) = 214635$ in the previous section.

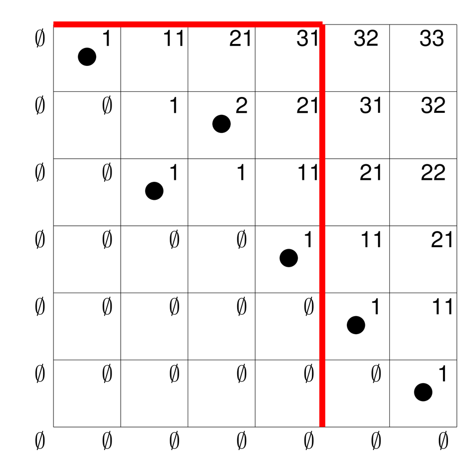
Definition: Let $\mathrm{seq}_k(\pi)$ be the sequence of partitions along the top and right borders of our board obtained by performing GDA_k.

Main Theorem: The sequence of partitions along the top and right borders of $\mathrm{GDA}(\phi^*(\pi))$ is the same as that in $\mathrm{GDA}_k(\pi)$.

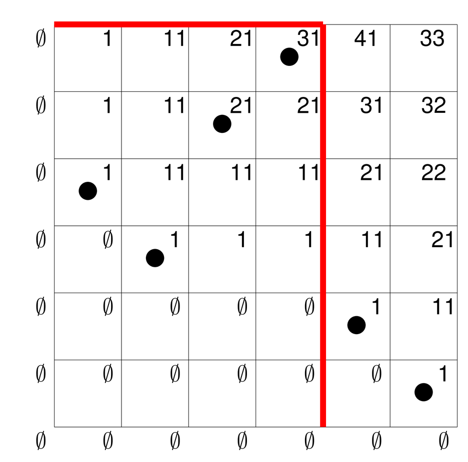
Commutativity Result: The fact that $\phi^*(\pi^{-1}) = \phi^*(\pi)^{-1}$ is made clear by our main theorem. This is because both GDA and GDA_k clearly commute with inverses.

5. Proof: An Idea

From the two boards below we first see that seq_k is invariant under applications of our map ϕ. The proof of this observation is long and somewhat technical. As a result it occupies most of the paper. Modulo this result the proof of the main theorem is a straightforward induction that is outlined below.



$\pi = 645321$



$\phi(\pi) = 435621$

Outline of Induction:

1. The red box is the smallest region containing markers moved by ϕ.
2. *The partitions along the red lines are the same. Therefore we have

$$\mathrm{seq}_k(\pi) = \mathrm{seq}_k(\phi(\pi)).$$

3. By an inductive argument we conclude that

$$\mathrm{seq}_k(\pi) = \mathrm{seq}_k(\phi(\pi)) = \mathrm{seq}(\phi^*(\pi)).$$

6. Primary References:

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