Patterns in Matchings and Rook Placements

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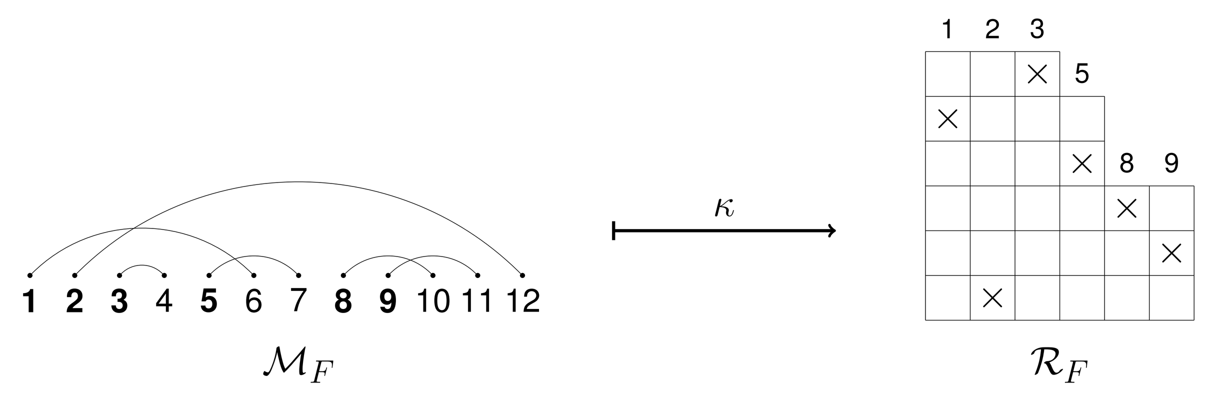
1. Matchings & Rook Placements: A Fundamental Bijection

Denote by $\mathcal{M}_n$ the set of perfect matchings on $[2n]$. Now consider the bijection κ to full rook placements on Ferrers boards:



where the NE border of the Ferrers board F is determined by (reading from left to right):

An opener is matched to a closer iff an $\times$ is placed in the corresponding row and column.

2. Pattern Avoidance Rook Placements

Let $\mathcal{R}_F(\sigma)$ be all rook placements on F that avoid σ.

Definition: If $|\mathcal{R}_F(\sigma)| = |\mathcal{R}_F(\tau)|$ for all Ferrers boards F, then σ and τ are *shape-Wilf-equivalent* and we write $\sigma \sim \tau$.

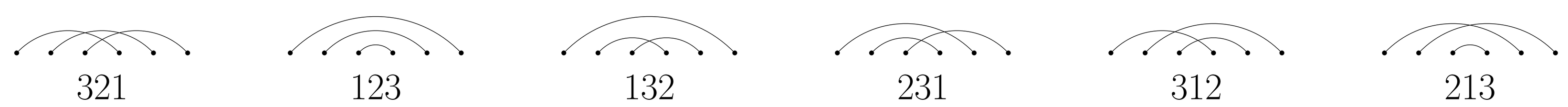
3. Pattern Avoidance in Matchings and Partitions

Matchings: The set of matchings that avoid σ is

$$\mathcal{M}_F(\sigma) := \kappa^{-1}(\mathcal{R}_F(\sigma)),$$

and $\mathcal{M}_n(\sigma) = \bigcup_{F \in \mathcal{F}_n} \mathcal{M}_F(\sigma)$, where the union is over all Ferrers boards with n rows and columns.

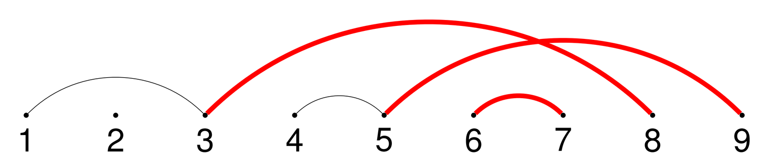
The patterns of length 3 are...



Observe:

k – noncrossing $\longleftrightarrow$ avoiding $k \ldots 21$ $\qquad$ k – nonnesting $\longleftrightarrow$ avoiding $12 \ldots k$

Set Partitions:
Let $\mathcal{P}_n$ denote the set of partitions of $[n]$. Pattern avoidance is defined in terms of patterns of arcs:



Contains 213

Denote by $\mathcal{P}_n(\sigma)$ the set of partitions of $[n]$ that avoid σ.

Observe: Fixed points never contribute to the formation of a pattern.

4. Patterns of Length 3: A Short Overview

Equivalence Classes:
S_3 breaks up into 3 shape-Wilf-equivalence classes (see [1, 6, 3, 8]):

$$231 \sim 312 \qquad 123 \sim 321 \sim 213 \qquad 132$$

Further, Stankova showed $|\mathcal{R}_F(231)| \le |\mathcal{R}_F(123)| \le |\mathcal{R}_F(132)|$.

Known Enumerative Results:

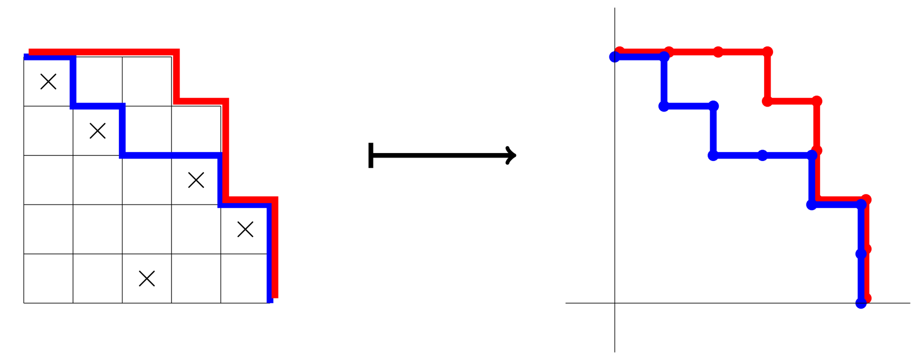
$$|\mathcal{M}_n(213)| = |\mathcal{D}_n^2|$$

where $\mathcal{D}_n^2$ = pairs of noncrossing Dyck paths. The previous proofs of this fact are long and complicated! See [1, 5, 7, 6].

5. A Simple Bijection between $\mathcal{M}_n(213)$ and D_n^2

Theorem 1 (J. Bloom, S. Elizalde). *There exists an explicit (and painfully simple!) bijection*

$$\mathcal{R}_n(213) \to \mathcal{D}_n^2.$$



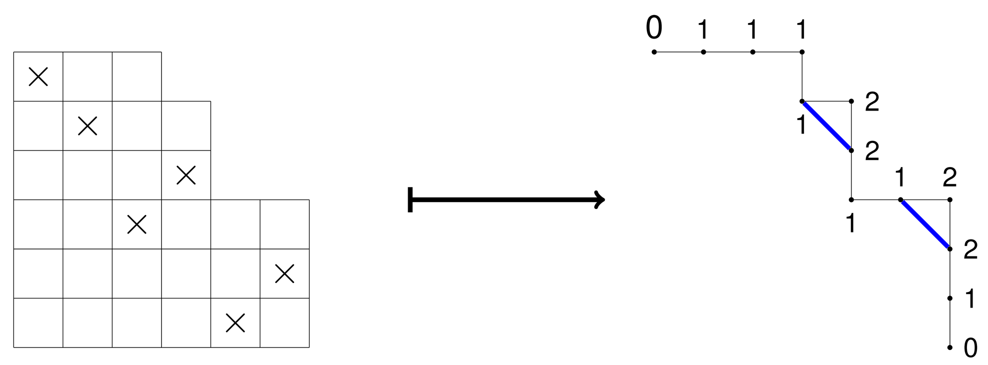
- **RED** path represents given Ferrers board
- **BLUE** path represents smallest Ferrers board

6. Enumeration of $\mathcal{M}_n(231)$

Theorem 2 (J. Bloom, D. Saracino). *There exists an explicit bijection*

$$\Pi : \mathcal{R}_F(231) \to \mathcal{L}_F,$$

where $\mathcal{L}_F$ is a set of "special" labelings of the border of F.



The label = length of the longest increasing sequence southwest of each vertex.

Defining Properties of $\mathcal{L}_F$:

- Monotone Property
- Zero Condition
- Diagonal Property

Theorem 3 (Bloom,Elizalde). *The generating function for 231-avoiding matchings is*

$$\sum_{n \ge 0} |\mathcal{M}_n(231)| z^n = \sum_{n \ge 0} |\mathcal{L}_n| z^n = \frac{54z}{1 + 36z - (1 - 12z)^{3/2}},$$

where $\mathcal{L}_n = \bigcup_{F \in F_n} \mathcal{L}_F$. The asymptotic behavior of its coefficients is given by

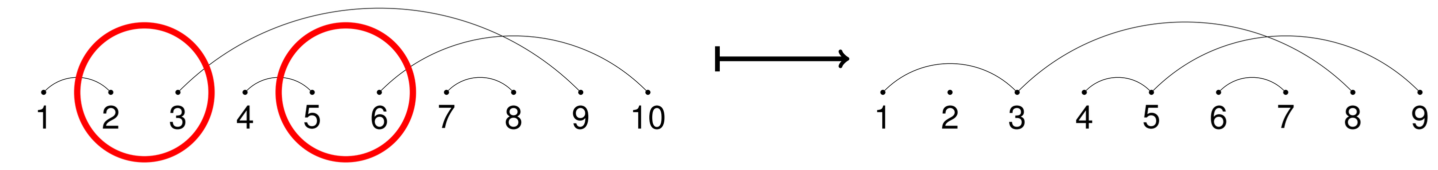
$$|\mathcal{M}_n(231)| \sim \frac{3^3}{2^5 \sqrt{\pi n^5}} 12^n.$$

Observe:

- Generating function for 231-avoiding matchings is **algebraic**
- Generating function for 123-avoiding matchings is **D-Finite** [5]

7. Enumeration of $\mathcal{P}_n(231)$

Set Partitions from Matchings:



Choosing to merge "valleys" and add fix points translates into...

$$\sum_{n \ge 0} |\mathcal{P}_n(\tau)| z^n = \frac{1}{1-z} A\left(\frac{1}{z}, \frac{z^2}{(1-z)^2}\right),$$

where

$$A(v, z) = \sum_{n \ge 0} \sum_{M \in \mathcal{M}_n(\tau)} u^{\mathrm{val}(M)} z^n.$$

Theorem 4 (Bloom, Elizalde). *The generating function $\sum_{n \ge 0} |\mathcal{P}_n(231)| z^n$ for 231-avoiding partitions is a root of the cubic polynomial*

$$(z-1)(5z^2 - 2z + 1)^2 B^3 + (-9z^5 + 54z^4 - 85z^3 + 59z^2 - 14z + 3) B^2 + (-9z^4 + 60z^3 - 64z^2 + 13z - 3) B + (-9z^3 + 23z^2 - 4z + 1).$$

The asymptotic behavior of its coefficients is given by

$$|\mathcal{P}_n(231)| \sim \delta n^{-5/2} \rho^n,$$

where $\delta \approx 0.061518$ and

$$\rho = \frac{3(9 + 6\sqrt{3})^{1/3}}{2 + 2(9 + 6\sqrt{3})^{1/3} - (9 + 6\sqrt{3})^{2/3}} \approx 6.97685$$

Observe:

- Generating function for 231-avoiding partitions is **algebraic**
- Generating function for 123-avoiding partitions is **D-Finite** [4]

8. Simultaneous Avoidance

Class	Matchings	Set partitions
{123,213}	$\frac{4}{3+\sqrt{1-8z}}$	$\frac{2-3z+z^2-z\sqrt{1-6z+z^2}}{2(1-3z+3z^2)}$
{123,231} & {123,312}	Solution of a cubic	Solution of a cubic
{123,321}	$\frac{1-5z+2z^2}{1-6z+5z^2}$	$\frac{1-10z+32z^2-37z^3+12z^4}{(1-z)(1-10z+31z^2-30z^3+z^4)}$
{213,321}	Functional equation	Unknown
{123,132} & {132,321}	Unknown	Unknown

9. Open Questions

- Find a generating function for 132-avoiding matchings and set partitions.

References

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