Final Exam Practice: solutions

1) a- Let (λ, v) be an eigen pair for N. As N is nilpotent we know $\exists\ m \geq 1$, so that $N^m = 0$.

Now
$$0 = N^m v = \lambda^m v.$$
As $v \neq 0_V$ (eigenvectors are never 0_V), then $\lambda = 0$.

b- The eigenspace corresp. to 0 for n is
$$\text{null}(T - 0) = \text{null}\, T.$$
By defn of a Jordan basis $N(N^k v) = 0$ so $N^k v \in \text{null}\, T$. We claim that
$$\text{null}\, T = \text{span}\,(N^k v).$$
so its dim is 1. To show this let $x \in \text{null}\, T$. As $\{v, Nv, \dots, N^k v\}$ is a basis then
$$x = a_0 v + a_1 N v + \dots + a_k N^k v.$$
Applying N to both sides, and using fact that $N^{k+1} v = 0$ we get:
$$0 = Nx = a_0 N v + \dots + a_{k-1} N^k v$$
As the $N^i v$ are linearly indep, then $a_0 = \dots = a_{k-1} = 0$
$\therefore\ x = a_k N^k v \in \text{span}\,(N^k v)$ as needed.

As 0 is the only eigenvalue, then $V = G_0$.
As J is a basis for $V = G_0$ then
$$\dim G_0 = \dim V = |J| = k+1$$

2) a- No, for ex we could have

$$\dim(\mathrm{Null}(T-2)) = 1$$

yet $\dim G_2 = 2$. (see HW 7 #4).

b- In this case, yes! The Jordan form for T in general can only be

$$A = \left[\begin{array}{cc|c} 2 & 0 & \\ 1 & 2 & \\ \hline & & 3 \end{array}\right] \text{ or } B = \left[\begin{array}{cc|c} 2 & 0 & \\ 0 & 2 & \\ \hline & & 3 \end{array}\right]$$

But $f(A) \neq 0$, so T must have Jordan form B, which is diag.

3) We know that $V = \mathcal{U} \oplus \mathcal{U}^\perp$. let $u, w \in V$

a- so $u = x_1 + y_1$ and $w = x_2 + y_2$ where $x_i \in \mathcal{U}$ and $y_i \in \mathcal{U}^\perp$. Now if $a, b \in \mathbb{F}$, then

$$au + bw = a(x_1 + y_1) + b(x_2 + y_2)$$
$$= \underbrace{(ax_1 + bx_2)}_{\in \mathcal{U}} + \underbrace{(ay_1 + by_2)}_{\in \mathcal{U}^\perp}.$$

Computing gives:

$$P(au + bw) = ax_1 + bx_2 = aP(u) + bP(w)$$

$\therefore P$ is linear.

(Note we write P instead of $P_{\mathcal{U}}$.)

b- Let $\{e_1 \dots e_k\}$ be an o.n. basis for $\mathcal{U}$. Extending gives an o.n. basis $\{e_1 \dots e_k, e_{k+1} \dots e_n\}$ for all of V. Now for $x \in V$ we have

$$x = \underbrace{\sum_{i=1}^{k} a_i e_i}_{Px \in \mathcal{U}} + \underbrace{\sum_{i=k+1}^{n} a_i e_i}_{\in \mathcal{U}^\perp}$$

Now as the e_i's are o.n. then the pythag. thm yields:

$$\|P_x\|^2 + \|x - P_x\|^2 = \|x\|^2$$

Therefore $\|Px\|^2 \le \|x\|^2 \Rightarrow \|Px\| \le \|x\|$.

c- $(\Rightarrow)$ If $\mathcal{U}$ is T-invariant, then if $w \in V$ w/ $w = x + y$ where $x \in \mathcal{U}$ + $y \in \mathcal{U}^\perp$ we have $TP(w) = Tx \in \mathcal{U}$ as $\mathcal{U}$ is T-inv.

As $Px' = x'$ for all $x' \in \mathcal{U}$ (why?), then

$$P(Tx) = Tx.$$

Together, we have $PTP(w) = TP(w)$ $\forall\, w \in V$.

$\therefore$ $PTP = TP$.

4) Let (λ, v) and (α, w) be eigen pairs where $\lambda \neq \alpha$. We consider two cases: Either one of λ or α is zero or neither are zero.

<u>case 1</u>: One of λ or α is zero.

wlog assum $\lambda = 0$ & $\alpha \neq 0$. Now

$$\langle v, w \rangle = \langle Uv, Uw \rangle = \langle 0_v, Uw \rangle = 0$$

so $v \perp w$.

<u>case 2</u>: Neither λ or α is zero.

In this case $\lambda \cdot \bar{\alpha} \neq 0$ and $\lambda \cdot \bar{\alpha} \neq 1$ (why!)

computing gives:

$$\langle v, w \rangle = \langle Uv, Uw \rangle = \langle \lambda v, \alpha w \rangle = \lambda \bar{\alpha} \langle v, w \rangle.$$

This can only occur provided $\langle v, w \rangle = 0$. $\therefore$ $v \perp w$.

5)