Practice problems #2: solutions

1) Let $\mathcal{B} = \{v_1, \dots, v_k\}$ be a basis for $\mathcal{U}$. We claim that

$$\{Tv_1, \dots, Tv_k\}$$

is a basis for $T\mathcal{U}$. This certainly implies the result.

Spanning

$$\begin{aligned}
\operatorname{span}(Tv_1, \dots, Tv_k) &= \{a_1 Tv_1 + \cdots + a_k Tv_k \mid a_i \in \mathbb{F}\} \\
&\overset{\text{linearity}}{=} \{T(a_1 v_1 + \cdots + a_k v_k) \mid a_i \in \mathbb{F}\} \\
&\overset{\mathcal{B} \text{ spans } \mathcal{U}}{=} \{Tu \mid u \in \mathcal{U}\} \\
&= T(\mathcal{U}).
\end{aligned}$$

Independence

Assume $a_i \in \mathbb{F}$ are such that

$$a_1 Tv_1 + \cdots + a_k Tv_k = 0_W$$

then $T(a_1 v_1 + \cdots + a_k v_k) = 0_W$

so $a_1 v_1 + \cdots + a_k v_k \in \operatorname{Null} T = \{0_V\}$, where the last equality follows since T is injective. So

$$a_1 v_1 + \cdots + a_k v_k = 0_V$$

$\Rightarrow a_i = 0$ as $\mathcal{B}$ is indep.

2) a) $[T]_{\mathcal{B}} = \left[[Te_1]_{\mathcal{B}} \ [Te_2]_{\mathcal{B}} \right] = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}_{\mathcal{B}} \right]$

$$= \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$$

b) By the change of basis thm

$$[T]_{\mathcal{E}} = [I_V]_{\mathcal{B}}^{\mathcal{E}} [T]_{\mathcal{B}}^{\mathcal{B}} [I_V]_{\mathcal{E}}^{\mathcal{B}}$$

Now
$$[I_V]_{\mathcal{C}}^{\mathcal{B}} = \left[\, \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}} \ \begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{B}} \,\right] = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

and
$$[I_V]_{\mathcal{B}}^{\mathcal{C}} = \left[\, \begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{C}} \ \begin{bmatrix}0\\1\end{bmatrix}_{\mathcal{C}} \,\right] = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$$

So
$$[T]_{\mathcal{C}} = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

<u>Note:</u> $[I_V]_{\mathcal{C}}^{\mathcal{B}} = \left([I_V]_{\mathcal{B}}^{\mathcal{C}}\right)^{-1}$ Must this occur always?

3) a) Let $x \in \mathcal{U} \cap W$. As $\mathcal{U}$ is T-invariant
$$Tx \in \mathcal{U}$$
Likewise, as W is T-invariant, then
$$Tx \in W.$$
$\therefore$ $Tx \in \mathcal{U} \cap W$. This means $\mathcal{U} \cap W$ is T-invariant.

b) Yes. To see why assume $\mathcal{U}$ is neither V nor $\{0_V\}$. Fix a basis
$$\{u_1, \dots, u_k\}$$
for $\mathcal{U}$. As $\mathcal{U}$ is not trivial $0 < k$. As $V \neq \mathcal{U}$, then $k \neq \dim V$. Now extend this to a basis
$$\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\}$$
for all of V. We will show $\mathcal{U}$ is not invariant for some operator on V.

Consider the operator $S : V \to V$ so that

$$Su_1 = v_1 \quad \text{and} \quad Sv_1 = u_1$$

but S fixes every other basis element. But now $u_1 \in \mathcal{U}$ but $Su_1 = v_1 \notin \mathcal{U}$. Therefore $\mathcal{U}$ cannot be S-invariant.

4) Assume $Sv \neq 0$, where (v, λ) is an eigenpair for T. Now

$$T(Sv) = TSv = STv = S\lambda v = \lambda(Sv)$$

$\therefore$ (λ, Sv) is an eigen pair as Sv is <u>not</u> the zero vector.

5) Consider the vector $v_1 + \cdots + v_n \in V$.

6) $(\Rightarrow)$ Assume T is diagonalizable. So $\exists$ a basis $\mathcal{B} = \{v_1 \cdots v_n\}$

so that

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

As $[Tv_i]_{\mathcal{B}} = [T]_{\mathcal{B}} [v_i]_{\mathcal{B}} = [T]_{\mathcal{B}} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ position

$$= \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ pos.}$$

So $Tv_i = \lambda_i v_i \Rightarrow (\lambda_i, v_i)$ is an eigen pair.

($\Leftarrow$) Now assume V has a basis

$$\beta = \{ v_1 \cdots v_n \}$$

that consists entirely of eigen vectors for T. If $\lambda_1, \cdots, \lambda_n$ are the corresponding eigen values, then

$$[T]_\beta = [\ [Tv_1]_\beta \cdots [Tv_n]_\beta\ ]$$

$$= [\ [\lambda_1 v_1]_\beta \cdots [\lambda_n v_n]_\beta\ ]$$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

so $[T]_\beta$ is diagonal.

For the last part, recall that if

$$\lambda_1, \cdots, \lambda_{\dim V}$$

are distinct eigenvalues for T w/ corresponding eigen vectors

$$x_1, \cdots, x_{\dim V}$$

then the x_i are indep. This means (as we have $\dim V$ of them) that

$$\beta = \{ x_1 \cdots x_{\dim V} \}$$

is a basis for V, that consists entirely of eigen vectors.

$\therefore$ T is diagonalizable by what we first proved.