<u>solutions: practice #1</u>

1) No. For example the dist law fails since

$$2[(1,1)+(1,1)] = 2(2,1) = (4,2)$$

but

$$2(1,1) + 2(1,1) = (4,4).$$

2) Let V be an n-dim space over $\mathbb{C}$. Then we can choose a basis

$$\mathcal{B} = \{v_1, \dots, v_n\}$$

for V (over $\mathbb{C}$). Now consider the set

$$S = \{v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n\}$$

of vectors in V where $i \in \mathbb{C}$. We claim that S is a basis for V as a space over $\mathbb{R}$.

For clarity, let us denote the vector space V over $\mathbb{C}$ as $V_{\mathbb{C}}$ and the space V over $\mathbb{R}$ as $V_{\mathbb{R}}$.

To show S is a basis for $V_{\mathbb{R}}$ we need to show it spans $V_{\mathbb{R}}$ and is indep (over $\mathbb{R}$).

<u>Span</u>

As $\mathcal{B}$ spans $V_{\mathbb{C}}$, then we have

$$\begin{aligned} V = \operatorname{span}_{\mathbb{C}}(\mathcal{B}) &= \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_i \in \mathbb{C}\} \\ &= \{(a_1 + ib_1)v_1 + \dots + (a_n + ib_n)v_n \mid a_i, b_i \in \mathbb{R}\} \\ &= \{a_1 v_1 + b_1 iv_1 + \dots + a_n v_n + b_n iv_n \mid a_i, b_i \in \mathbb{R}\} \\ &= \operatorname{span}_{\mathbb{R}}(S). \end{aligned}$$

So our set S spans V over $\mathbb{R}$.

Independence

Assume $a_j, b_j \in \mathbb{R}$ are such that

$$a_1 v_1 + b_1 i v_1 + \cdots + a_n v_n + b_n i v_n = 0_V$$

so $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0_V$ where $\alpha_j = a_j + i b_j \in \mathbb{C}$.

As the v_i's are indep over $\mathbb{C}$, we must have all $\alpha_j = 0$. Then all a_j and $b_j = 0$ too, as needed.

$\therefore$ S is indep over $\mathbb{R}$.

We conclude that $\dim_{\mathbb{R}} V = |S| = 2n$.

3) a- To see that $tr \in L(M_n, \mathbb{R})$ note that

$$tr\Big(\underbrace{\begin{bmatrix} a_1 & & * \\ & \ddots & \\ * & & a_n \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} b_1 & & * \\ & \ddots & \\ * & & b_n \end{bmatrix}}_{B}\Big) = tr\Big(\begin{bmatrix} a_1+b_1 & & * \\ & \ddots & \\ * & & a_n+b_n \end{bmatrix}\Big)$$

$$= a_1 + b_1 + \cdots + a_n + b_n$$

$$= tr(A) + tr(B).$$

Also $tr(sA) = sa_1 + \cdots + sa_n = s(a_1 + \cdots + a_n)$

$$= s \cdot tr(A).$$

so $tr \in L(M_n, \mathbb{R})$.

b- $\text{null}\, tr = \{A \in M_n \mid tr\, A = 0\}$. These are often called <u>traceless</u> matrices.

By rank-nullity and the observation that

$$\text{ran}\, tr = \mathbb{R},$$

we have

$$\dim(\text{null}\, tr) = \dim M_n - \dim(\text{ran}\, tr) = n^2 - 1.$$

4) First, we see zero poly $\in \mathcal{U}$. Now let $p, q \in \mathcal{U}$.

closure of addition

$$(p+q)(-x) = p(-x) + q(-x) = -p(x) - q(x) = -(p+q)(x)$$

so $p+q \in \mathcal{U}$

closure of scalar mult

Let $s \in \mathbb{F}$. Now

$$(sp)(-x) = sp(-x) = -sp(x) = -(sp)(x)$$

so $sp \in \mathcal{U}$.

we conclude that $\mathcal{U}$ is a subspace of $\mathcal{P}_{\leq n}$.

A similiar pf shows W is also a subspace of $\mathcal{P}_{\leq n}$.

b- observe that if n is even, then

$$\{x, x^3, \ldots, x^{n-1}\} \quad \text{and} \quad \{1, x^2, \ldots, x^n\}$$

are bases for $\mathcal{U}$ and W, resp. In this case

$$\dim \mathcal{U} = n/2 \quad \text{and} \quad \dim W = n/2 + 1.$$

If n is odd, then

$$\{x, x^3, \ldots, x^n\} \quad \text{and} \quad \{1, x^2, \ldots, x^{n-1}\}$$

are bases for $\mathcal{U}$ and W. In this case

$$\dim \mathcal{U} = \frac{n+1}{2} \quad \text{and} \quad \dim W = \frac{n+1}{2}.$$

c- To see that $\mathcal{U} \oplus W = \mathcal{P}_{\leq n}$, first observe that if $p \in \mathcal{U} \cap W$, then

$$p(x) = p(-x) = -p(x)$$

or $2p(x) = 0$. This means $p(x)$ is the zero poly $0_{\mathcal{P}_{\leq n}}$ and $\mathcal{U} \cap W = \{0_{\mathcal{P}_{\leq n}}\}$. Now we know the direct sum $\mathcal{U} \oplus W$ is a subspace of $\mathcal{P}_{\leq n}$.

To see that we have equality, observe that

$$\dim(\mathcal{U} \oplus W) \underset{\text{Hw \#3}}{=} \dim \mathcal{U} + \dim W \underset{\text{part b}}{=} n+1.$$

so $\dim(\mathcal{U} \oplus W) = \dim \mathcal{P}_{\leq n} \Rightarrow \mathcal{U} \oplus W = \mathcal{P}_{\leq n}$.

5) Let $\{v_1 \ldots v_k\}$ be a basis for null$T$. By the basis ext. theorem this set extends to

$$\{v_1 \ldots v_k, u_1 \ldots u_m\}$$

a basis for all of V. Now define

$$\mathcal{U} = \operatorname{span}(u_1 \ldots u_m)$$

Note $\dim(\text{null}T) = k$, $\dim \mathcal{U} = m$, and $\dim V = m+k$.

Moreover observe that

$$\mathcal{U} \cap \text{null}T = \{0_V\}$$

why? So $\mathcal{U} \oplus \text{null}T$ is a subspace of V.

As $\dim(\mathcal{U} \oplus \text{null}T) = m+k = \dim V$ we see that

$$\mathcal{U} \oplus \text{null}T = V.$$

Lastly observe that

$$\begin{aligned}
\text{ran}\,T = \{Tv \mid v \in V\} &= \{T(u+w) \mid u \in \mathcal{U}, w \in \text{null}T\} \\
&= \{Tu + \cancel{Tw}^{\,0} \mid \text{''} \quad \text{''}\} \\
&= \{Tu \mid u \in \mathcal{U}\} = T(\mathcal{U}).
\end{aligned}$$

Pictorially we can view this as: