Solutions: Exam #1

1. a) No. For example, I is not an identity since

$$I(a,b) \neq (a,b)$$ in general.

b) No. $\mathcal{U}$ is <u>not</u> closed under addition.

It is however closed under scalar mult interestingly.

c) Let $\mathcal{U} = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ and

$W = \operatorname{span}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$. Clearly $\mathcal{U} \cap W = \{0_{\mathbb{R}^3}\}$

and $\mathcal{U} \oplus W = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \mathbb{R}^3$.

d) observe that

$$\operatorname{span} T = \{(x,y,z) \mid x+y+z=0\} = \left\{ \begin{bmatrix}x\\y\\-x-y\end{bmatrix} \mid x,y \in \mathbb{R}\right\}$$

$$= \left\{ x\begin{bmatrix}1\\0\\-1\end{bmatrix} + y\begin{bmatrix}0\\1\\-1\end{bmatrix} \mid x,y \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$$

As $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ is clearly indep. then $\dim \operatorname{span} T = 2$.

By rank nullity $\dim \operatorname{null} T = \dim \mathbb{R}^3 - \dim \operatorname{span} T$

$$= 3 - 2 = 1.$$

We conclude that T is neither surjective nor injective.

2) As f has degree exactly n, then $f^{(i)}$ has degree exactly $n-i$. Now assume we have scalars $a_0, \dots, a_n$ so that

$$0 = a_0 f + a_1 f^{(1)} + \dots + a_n f^{(n)}.$$

As f is the only poly in $\mathcal{B}$ w/ an x^n term then $a_0 = 0$. so we must have:

$$0 = a_1 f^{(1)} + \dots + a_n f^{(n)}.$$

Among the remaining poly $f^{(1)}$ is the only one w/ an x^{n-1} term, so $a_1 = 0$. Continuing in this fashion we see that $a_0 = a_1 = \dots = a_n = 0$.

$\therefore$ $\mathcal{B}$ is indep.

As $|\mathcal{B}| = n+1 = \dim(\mathcal{P}_{\leq n})$, we can conclude by thm. 2.10 that $\mathcal{B}$ is a basis.

b- as $\mathcal{B}$ is a basis it spans $\mathcal{P}_{\leq n}$. so for any $g \in \mathcal{P}_{\leq n}$ this means $\exists$ scalars $c_0, \dots, c_n \in \mathbb{F}$ so that

$$g = c_0 f + c_1 f^{(1)} + \dots + c_n f^{(n)}.$$

3) As S is dep we know $\exists$ scalars, not all zero $a_\phi \dots a_n$ so that

$$a_1(v_1+u) + a_2(v_2+u) + \dots + a_n(v_n+u) = 0$$

or equivalently

$$a_1 v_1 + \dots + a_n v_n + (a_1 + \dots + a_n) u = 0.$$

Note that $a_1 + \dots + a_n \neq 0$. If it were then we would have

$$a_1 v_1 + \dots + a_n v_n = 0$$

w/ a_i's not all zero. This contradicts the fact that L is indep.

$$\therefore \quad u = \frac{1}{a_1 + \dots + a_n} (a_1 v_1 + \dots + a_n v_n) \in \text{span}(L).$$

4) As $T: V \to W$ is nonzero and $\dim W = 1$ then T is surjective. By rank-nullity

$$\begin{aligned} \dim \text{null}\, T &= \dim V - \dim \text{ran}\, T \\ &= n - 1. \end{aligned}$$

Observe that if $u \notin \text{null}\, T$, then

$$\text{span}(u) \cap \text{null}\, T = \{0_V\}.$$

so $\text{span}(u) \oplus \text{null}\, T \subseteq V$. As

$$\begin{aligned} \dim V = n = 1 + (n-1) &= \dim(\text{span}\, u) + \dim \text{null}\, T \\ &= \dim(\text{span}(u) \oplus \text{null}\, T) \end{aligned}$$

we conclude that

$$V = \text{span}\, u \oplus \text{null}\, T.$$