Solutions #9

1) Computing we see that

$$\|u+v\|^2 - \|u-v\|^2 = \langle u+v, u+v\rangle - \langle u-v, u-v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$- (\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle)$$
$$= 4\langle u,v\rangle \checkmark$$

Solving gives the desired expression for $\langle u,v\rangle$.

2) First, assume $v \in \text{span}(e_1, \ldots, e_k)$. Then we know from Thm 6.17 (book) we know

$$\|v\|^2 = |\langle v,e_1\rangle|^2 + \cdots + |\langle v,e_k\rangle|^2.$$

For the other direction, first extend $\{e_1 \ldots e_k\}$ to a o.n basis $\{e_1 \ldots e_k \ldots e_n\}$ for all of V. This is possible by Corollary 6.25 (book). Now by thm 6.17 we have

$$|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_k\rangle|^2 = \|v\|^2 = |\langle v,e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2.$$

It now follows that

$$|\langle v, e_{k+1}\rangle| = \cdots = |\langle v, e_n\rangle| = 0.$$

which implies that $\langle v, e_{k+1}\rangle = \cdots = \langle v, e_n\rangle = 0$.

Therefore:

$$v = \langle v,e_1\rangle e_1 + \cdots + \langle v, e_k\rangle e_k + \cdots + \langle v, e_n\rangle e_n$$
$$= \langle v,e_1\rangle e_1 + \cdots + \langle v, e_k\rangle e_k \in \text{span}(e_1 \ldots e_k).$$

con't) The converse is true. Let $\{f_1 \cdots f_n\}$ be an o.n. basis for $\mathbb{C}^n$ and consider the matrix $A = [f_1 \cdots f_n]$. Now fix

$$v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

in $\mathbb{C}^n$. Computing shows

$$\langle Av, Aw \rangle = \left\langle \sum_{i=1}^n a_i f_i , \sum_{j=1}^n b_j f_j \right\rangle$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{b_j} \langle f_i, f_j \rangle \quad \left(\text{by } \begin{matrix}\text{linearity \&} \\ \text{conj lin}\end{matrix}\right)$$

$$= \sum_{i=1}^n a_i \overline{b_i} \langle f_i, f_i \rangle \quad (\text{As } f_i \text{ are orthog}).$$

$$= \sum_{i=1}^n a_i \overline{b_i} \quad (\text{as } \|f_i\| = 1)$$

$$= \langle v, w \rangle \quad (\text{as } \langle , \rangle \text{ is dot prod}).$$

$\therefore$ A is unitary.

3) a- Computing shows that

$$\|\mathcal{U}v\|^2 = \langle \mathcal{U}v, \mathcal{U}v\rangle = \langle v, v\rangle = \|v\|^2.$$

so $\mathcal{U}$ preserves length.

b- Let (λ, v) be an eigenpair for $\mathcal{U}$. Now

$$\underbrace{\|v\| = \|\mathcal{U}v\|}_{\text{by a)}} = \|\lambda v\| = |\lambda|\,\|v\|.$$

As $v \neq 0_V$, then $\|v\| \neq 0$, so we can conclude that

$$|\lambda| = 1.$$

$\therefore$ All eigenvalues for $\mathcal{U}$ lie on the unit circle in $\mathbb{C}$.

c- We know from Hw #6 problem #1, that

T is invertible iff 0 is not an eigenvalue.

From b) we know 0 is not an eigenvalue, hence T is invertible.

d- Let $A = [v_1, \ldots, v_n]$, so that v_i is the i^{th} column of A. Now let $e_1 \ldots e_n$ be the standard basis vectors for $\mathbb{C}^n$. Now

$$\langle v_i, v_j\rangle = \langle Ae_i, Ae_j\rangle = \underbrace{\langle e_i, e_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j.\end{cases}}_{\text{as the inner product is dot prod.}}$$

Therefore the columns of A are an orthonormal set, of size n. As this implies $\{v_1 \ldots v_n\}$ is an indep. set of size n, we conclude the cols form an o.n. basis for $\mathbb{C}^n$.

4) First, recall that

$$V = \operatorname{ran} P \oplus (\operatorname{ran} P)^{\perp}.$$

Consequently,

$$\dim \operatorname{ran} P^{\perp} = \underbrace{\dim V - \dim \operatorname{ran} P.}_{\dim \operatorname{null} P \text{ (by rank-nullity)}}$$

As $\operatorname{null} P \subseteq \operatorname{ran} P^{\perp}$, then

$$\dim \operatorname{null} P \leq \dim \operatorname{ran} P^{\perp} = : \dim \operatorname{null} P.$$

This implies $\operatorname{null} P = \operatorname{ran} P^{\perp}$ as desired. This establishes the fact that

$$V = \operatorname{ran} P \oplus \operatorname{null} P.$$

Let $v = x + y$ where $x \in \operatorname{ran} P$ & $y \in \operatorname{null} P$.
To show P is the orthogonal proj onto $\operatorname{ran} P$ we must show $Pv = x$. To do this observe that as $x \in \operatorname{ran} P$, then $\exists$ some $x' \in V$ so that $x = Px'$. Now

$$Pv = P(x+y) = P(Px'+y) = P^2x' + \cancel{Py}^{\to 0}$$
$$= Px' = x.$$

This completes the proof.