Solutions #8

1) Assume $(ST)^k = 0$. Now

$$(TS)^{k+1} = \underbrace{(TS)(TS)\cdots(TS)}_{k+1} = T\underbrace{(ST)\cdots(ST)}_{k}S = 0.$$

Therefore TS is also nilpotent.

2) Assume for a contradiction that S is dependent. This means $\exists\ a_i \in \mathbb{F}$, not all zero, $\ni$

$$a_0 v + a_1 N v + \cdots + a_k N^k v = 0_V.$$

Let i be the smallest index so that $a_i \neq 0$. Consequently, $a_0, \ldots, a_{i-1} = 0$. As N is nilpotent and $N^k v \neq 0$, there must be some $m > k$ so that

$$N^{m-1} v \neq 0_V \quad \text{but} \quad N^m v = 0_V.$$

Now we have

$$\begin{aligned} 0_V &= N^{m-i-1}(a_0 v + a_1 N v + \cdots + a_k N^k v) \\ &= N^{m-i-1}(a_i N^i v + \cdots + a_k N^k v) \\ &= a_i N^{m-1} v + \underbrace{a_{i+1} N^m v + \cdots + a_k N^{k+m-i-1} v}_{=0_V} \end{aligned}$$

Therefore $0_V = a_i N^{m-1} v$. As $N^{m-1} v \neq 0_V$, then $a_i = 0$. This contradicts our choice of i. So S is indep as claimed.

3) up to rearrangements of Jordan blocks, we have only the following 5:

$$\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} \boxed{\begin{matrix} 2 & \\ 1 & 2 \end{matrix}} & & \\ & 2 & \\ & & 2 \end{bmatrix}, \begin{bmatrix} \boxed{\begin{matrix} 2 & \\ 1 & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & \\ 1 & 2 \end{matrix}} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix} 2 & & \\ 1 & 2 & \\ & 1 & 2 \end{matrix}} & \\ & 2 \end{bmatrix}, \begin{bmatrix} 2 & & & \\ 1 & 2 & & \\ & 1 & 2 & \\ & & 1 & 2 \end{bmatrix}$$

Inner product spaces

1) a- Let $f, g, h \in \mathcal{C}[0,1]$. Now

$$\langle f, f \rangle = \int_0^1 f^2 \, dx \geq 0 \quad \text{w/ equality iff } f = 0.$$

so $\langle , \rangle$ is positive-definite. Also, we have

$$\langle af + bg, h \rangle = \int_0^1 (af + bg) h \, dx$$

$$= a\int_0^1 fh + b\int_0^1 gh \, dx \quad \left(\text{Linearity of integrals}\right)$$

$$= a\langle f, h \rangle + b\langle g, h \rangle.$$

so $\langle , \rangle$ is linear in the 1st coord. Lastly,

$$\langle f, g \rangle = \int_0^1 fg \, dx = \int_0^1 gf \, dx = \langle g, f \rangle.$$

so $\langle , \rangle$ is symmetric. Note, as $\mathcal{C}[0,1]$ is a real vector space then symmetry and conj. symmetry are equivalent.

This shows $\langle , \rangle$ is an inner product on $\mathcal{C}[0,1]$.

b) $\|x^n\| = \sqrt{\langle x^n, x^n\rangle} = \left(\int_0^1 x^{2n}\,dx\right)^{1/2} = \sqrt{1/(2n+1)}$.

c) To show $\cos\pi x \perp \sin\pi x$ we must check that $\langle \cos\pi x, \sin\pi x\rangle = 0$. Now

double ∡ formula.

$$\langle \cos\pi x, \sin\pi x\rangle = \int_0^1 \cos\pi x \cdot \sin\pi x\,dx = \frac{1}{2}\int_0^1 \sin 2\pi x\,dx$$

$$= 0.$$

2) b- Let $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$.

Now

$$\langle v, v\rangle = AX \cdot X$$

where $X = \begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{bmatrix}$, by the observation in a). As A is positive-definite, we see that $\langle , \rangle$ is too. The fact that $\langle , \rangle$ is linear in the 1^{st} coord. is clear from the defn. Lastly we must check conjugate symmetry:

$$\overline{\langle v, w\rangle} = \overline{\sum_{i=1}^n \sum_{j=1}^n a_i \bar{b}_j \langle v_i, v_j\rangle}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \bar{a}_i b_j \overline{\langle v_i, v_j\rangle}$$

$$= \sum_{j=1}^n \sum_{i=1}^n b_j \bar{a}_i \langle v_j, v_i\rangle \quad (\text{as } A^* = A).$$

$$= \langle w, v\rangle.$$

Together, these properties imply that $\langle , \rangle$ is an inner product.

c) If $A = I_n$, then

$$\langle v_i, v_j \rangle = A_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Consequently $\mathcal{B}$ is orthonormal.

d) If you fix a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ and let A be the $n \times n$ matrix obtained by setting its (i,j)-entry to be $\langle v_i, v_j \rangle$, then A will be positive-definite. First $A^* = A$ since

$$\overline{\langle v_i, v_j \rangle} = \langle v_j, v_i \rangle$$

and

$$Ax \cdot x > 0 \quad \text{provided } x \neq 0 \in \mathbb{R}^n$$

since

$$\langle u, u \rangle > 0 \quad \text{provided } u \neq 0_V \in V.$$