Solutions: HW #7

1) ($\Rightarrow$) Assume $\mathcal{U} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_m$ and let $u \in \mathcal{U}$. To prove this direction, assume

$$w_1 + \cdots + w_m = u = v_1 + \cdots + v_n$$

where $w_i, v_i \in \mathcal{U}_i$. Therefore

$$(u_1 - v_1) + \cdots + (u_m - v_m) = u - u = 0_V \in \mathcal{U}.$$

By defn of a direct sum, we conclude that $u_i - v_i = 0_V$ or $u_i = v_i$.

($\Leftarrow$) The assumptions imply that the only choice of $u_i \in \mathcal{U}_i$ $\ni$ $0_V = u_1 + \cdots + u_m$ is when $u_i = 0_V$. Therefore $\mathcal{U}$ is a direct sum of the $\mathcal{U}_i$.

2) Assume $T^n = 0$. If (λ, v) is an eigenpair for T, then

$$0_V = T^n v = \lambda^n v.$$

As $v \neq 0_V$, then $\lambda^n = 0 \Rightarrow \lambda = 0$.

($\Leftarrow$) If the only eigenvalue for T is 0, then its characteristic poly must be

$$P_T(x) = x^n$$

where $n = \dim V$. By Cayley-Hamilton we see that

$$0 = P_T(T) = T^n.$$

3) We know that T is invertible iff 0 is not an eigenvalue for T. As the characteristic poly for T is

$$P_T(x) = (x-\lambda_1)\cdots(x-\lambda_n)$$

where λ_i is an eigen value for T, we see that

$P_T(x)$ has a non-zero const. term $\Leftrightarrow$ each $\lambda_i \neq 0$.

Putting these two observations together, we obtain

T is invertible iff $P_T(x)$ has nonzero const. term

iff $P_T(0) \neq 0$.

b) By a) we know that if T is invertible, then

$$P_T(x) = x^n + \cdots + a_1 x + a_0$$

where $a_0 \neq 0$. By Cayley-Hamilton we know further that

$$0 = P_T(T) = T^n + \cdots + a_1 T + a_0 \cdot I$$

so $I = \frac{1}{a_0}\left(T^n + \cdots + a_1 T\right) = \underbrace{\frac{1}{a_0}\left(T^{n-1} + \cdots + a_1\right)}_{q(T).} T$

$\therefore$ $q(T) \cdot T = I = T\, q(T)$. In other words,

$$T^{-1} = q(T).$$

4) a) This follows from the fact that

$$\text{null}(T-\lambda_i)^n = G_i.$$

b) ($\Rightarrow$) Assume G_i has a basis consisting entirely of eigenvectors, say $\mathcal{B} = \{v_1 \dots v_k\}$. so

$$G_i = \text{span}(\mathcal{B}) \subseteq \text{null}(T-\lambda_i) \subseteq G_i$$

$$\therefore \quad G_i = \text{null}(T-\lambda_i)$$

($\Leftarrow$) This is obvious.

c) ($\Leftarrow$) Assume the alg mult. (a.e.) is equal to the geo mult. (g.e.) for each λ_i. Therefore

$$\dim(\text{null}(T-\lambda_i)) = \dim G_i$$

so by a) $\text{null}(T-\lambda_i) = G_i$. By thm 4.15 we see that

$$V = G_1 \oplus \dots \oplus G_m = \text{null}(T-\lambda_1) \oplus \dots \oplus \text{null}(T-\lambda_m)$$

where $\lambda_1 \dots \lambda_m$ are the distinct eigenvalues. If $\mathcal{B}_i$ is a basis (of eigenvectors) for each $\text{null}(T-\lambda_i)$ then $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$ is a basis for V consisting entirely of eigenvectors for T.

$\therefore$ T is diagonalizable.

($\Rightarrow$) Assume T is diagonalizable. Therefore there must exist a basis $\mathcal{B}$ consisting entirely of eigenvectors for T. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues for T. Let

$$\mathcal{B}_i = \{ v \in \mathcal{B} \mid Tv = \lambda_i \}.$$

Certainly, $\mathcal{B}_i \subseteq \text{null}(T-\lambda_i)$. Therefore

$$|\mathcal{B}_i| \leq \dim(\text{null}(T-\lambda_i)) \leq \dim G_i.$$

As $V = G_1 \oplus \cdots + G_m$ and

$$n = \dim V = \sum_{i=1}^{m} \dim G_i \geq \sum_{i=1}^{m} |\mathcal{B}_i| = |\mathcal{B}| = n.$$

It now follows that $\mathcal{B}_i = \dim G_i$. In other words, the g.m. for λ_i equals its a.m..