Solutions #6

1) a- T is invertible iff T is injective

iff null $T = \{0_V\}$

iff null $(T - 0) = \{0_V\}$

iff 0 is not an eigen vector.

b- By symmetry, it will suffice to show that if λ is an eigen value for T, then λ^{-1} is one for T^{-1}. Now let x be an eigenvector corresp. to λ. So

$$Tx = \lambda x.$$

Mult. both sides by T^{-1} gives

$$\underbrace{T^{-1}T}_{I_V} x = \lambda T^{-1} x$$

This becomes $x = I_V x = \lambda T^{-1} x$ or

$$\lambda^{-1} x = T^{-1} x.$$

This proves the claim. (Note: we used part a) to ensure $\lambda \neq 0$ so that λ^{-1} exists!!).

2) For a contradiction, assume T has (at least) $k+2$ distinct eigen values.. This means we have at least $k+1$ distinct non zero eigenvalues. Let

$$\lambda_1, \dots, \lambda_{k+1}$$

be these values w/ corresponding eigen vectors

$$x_1, \dots, x_{k+1}.$$

By thm 4.4, these are linearly indep. It will now suffice to show $x_i \in \operatorname{ran} T$. (why?)

To do this observe that

$$\lambda_i x_i = T x_i \in \operatorname{ran} T.$$

As $\operatorname{ran} T$ is closed under scalar mult. and λ_i^{-1} exists ($\lambda_i \neq 0$), then

$$x_i = \lambda_i^{-1} \lambda_i x \in \operatorname{ran} T$$

as desired.

3) we wish to find all $x, y, z, \lambda \in \mathbb{C}$ so that

$$\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ 0 \\ 5z \end{bmatrix}.$$

This gives 3 equations:

$$\lambda x = 2y, \quad \lambda y = 0, \quad \lambda z = 5z.$$

$\underline{\lambda = 0}$

In this case, $z = 0$. Also $y = 0$, but x is free.

so $\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$ are the corresp. vectors.

$\underline{\lambda = 5}$

Then $y = 0$, and in turn, $x = 0$. so z is free and

$\left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R} \right\}$ are the corres. eigenvect.

These are the only solutions, since

$$\lambda x = 2y \Rightarrow \lambda^2 x = 2\lambda y = 0$$

so $\lambda = 0$ (first case) or $x = 0$ and $\lambda \neq 0$. Then $y = 0$ and the only way to have a non zero eigenvector is for $z \neq 0 \Rightarrow \lambda = 5$.

b) Consider the mapping

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} -b \\ a \\ -d \\ c \end{bmatrix}$$

on $\mathbb{R}^4$. The same proof as the rotation example given in class works here.

4) Let (λ, x) be an eigenvalue for ST.

<u>Case 1:</u> $Tx \neq 0_V$

Then $STx = \lambda x$, and mult both sides by T yields

$$TS(Tx) = \lambda(Tx)$$

so (λ, Tx) is an eigenpair for TS.

<u>Case 2:</u> $Tx = 0_V$

As $Tx = 0_V$, then $STx = S0_V = 0_V = \lambda x$ so $\lambda = 0$ in this case. We must show $\lambda = 0$ is an eigenvalue for TS. Now

0 is e.v. for TS iff TS is not injective " $(*)$
iff TS is not surjective.

As T is not injective (why?) then it cannot be surjective. Therefore TS cannot be surjective. We conclude from $(*)$ that 0 is an eigen value for TS.

5) a- Let $v \in V$ be non zero. Assume it is not an eigenvector.
Extend it to de basis $\{v_1, \dots v_n\}$ where $v_1 = v$.
As v is not an eigenvector then
$$Tv = a_1 v_1 + \dots + a_n v_n$$
and some $a_i \neq 0$ for $i > 1$. w/og assume $a_n \neq 0$.
Now consider the $n-1$ dim space
$$U = \text{span}(v_1, \dots, v_{n-1}).$$
By what we just showed U is not T-invariant
$\therefore$ Every nonzero vector is a T-eigenvector.

b- Assume $\lambda \neq \alpha$ are eigenvalues, w/ corresp vectors u, v. Then $T(u+v) = \lambda u + \alpha v$.
since $u+v \neq 0_V$, then this shows $u+v$ is not an eigenvector, $\Rightarrow\Leftarrow$.