Solutions: HW #5

1) a- In this case

$$[T]_{\mathcal{B}}^{e} = \left[[Te_1]_e \quad [Te_2]_e \right]$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}$$

b- First note that

$$Te_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$Te_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

so $[T]_{\mathcal{B}}^{e} = \left[[Te_1]_e \quad [Te_2]_e \right]$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ \tfrac{1}{2} & 0 \end{bmatrix}$$

c) Directly we get $T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

using the matrix from b (it is more interesting than the one in a) we see that

$$\left[T\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right]_e = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ \tfrac{1}{2} & 0 \end{bmatrix} \left[\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right]_{\mathcal{B}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ \tfrac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ \tfrac{3}{2} \end{bmatrix}$$

This does agree as

$$T\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \tfrac{3}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

2) a- To compute this directly, note that

$$ST\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = S\left(2a + (a+b)x + (a+c)x^2 + (a+d)x^3\right)$$

$$= a+b + 2(a+c)x + 3(a+d)x^2.$$

so $$[ST]_{\beta}^{\gamma} = \left[[STe_1]_{\gamma} \ [STe_2]_{\gamma} \ [STe_3]_{\gamma} \ [STe_4]_{\gamma} \right]$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$$

b- $$[S]_{\mathcal{C}}^{\gamma} = \left[[S\cdot 1]_{\gamma} \ [Sx]_{\gamma} \ [Sx^2]_{\gamma} \ [Sx^3]_{\gamma} \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$[T]_{\beta}^{\mathcal{C}} = \left[[Te_1]_{\mathcal{C}} \ [Te_2]_{\mathcal{C}} \ [Te_3]_{\mathcal{C}} \ [Te_4]_{\mathcal{C}} \right]$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

c) By thm 3.12

$$[ST]_{\beta}^{\gamma} = [S]_{\mathcal{C}}^{\gamma} \cdot [T]_{\beta}^{\mathcal{C}} =$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$$ (which agrees w/ a)).

3) As $\dim V = n = \dim \mathbb{R}^n$ it will suffice to prove T is injective by Corollary 3.6.

To see T is injective, let $v \in V$ be such that

$$Tv = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

As $Tv = [v]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, this means

$$v = 0v_1 + 0v_2 + \cdots + 0v_n = 0$$

where $\mathcal{B} = \{v_1, \dots v_n\}$. This shows $\operatorname{null} T = \{0_V\}$ and hence T is injective as needed

4) To start, choose a basis $\mathcal{B}' = \{u_1, \dots u_k\}$ for $\mathcal{U}$. Next, extend this basis to

$$\mathcal{B} = \{u_1, \dots u_k, v_1, \dots v_{n-k}\}$$

a basis for all of V. Now observe that as $Tu_i \in \mathcal{U}$ then

$$Tu_i = a_1 u_1 + \cdots + a_k u_k + 0v_1 + \cdots + 0v_{n-k}$$

Therefore

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \left[\, [Tu_1]_{\mathcal{B}} \cdots [Tu_k]_{\mathcal{B}}\ [Tv_1]_{\mathcal{B}} \cdots [Tv_{n-k}]_{\mathcal{B}} \,\right]$$

$$= k\left\{ \begin{bmatrix} A & * \\ O & * \end{bmatrix} \right.$$

k

last $n-k$ are arbitrary coluns as $[Tv_i]_{\mathcal{B}}$ is arbitrary

first k coluns have bottom $n-k$ entries all zeros as these coluns are just $[Tu_i]_{\mathcal{B}}$