Solutions: HW # 4

1) To show T is surjective, it will suffice to prove $\dim(\operatorname{ran} T) = 2 = \dim \mathbb{R}^2$.

To do this it will suffice to show

$$\dim(\operatorname{null} T) = 2$$

since rank-nullity will ensure that

$$2 = 4 - 2 = \dim \mathbb{R}^4 - \dim(\operatorname{null} T) = \dim \operatorname{ran} T.$$

To show $\dim \operatorname{null} T = 2$, we claim

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for null } T.$$

Certainly these vectors are indep. Moreover,

$$\operatorname{null} T = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a = 2d,\ b = 7c \right\}$$

$$= \left\{ \begin{bmatrix} 2d \\ 7c \\ c \\ d \end{bmatrix} \;\middle|\; c, d \in \mathbb{R} \right\} = \left\{ d\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; c, d \in \mathbb{R} \right\}$$

$$= \operatorname{span}(S).$$

2) a- As T is nonzero, then $\operatorname{ran} T = \mathbb{R}$. (why?) so

$$\dim(\operatorname{null} T) = \dim \mathbb{R}^3 - \dim(\operatorname{ran} T)$$
$$= 3 - 1 = 2.$$

b- consider the ~~the~~ standard basis $\left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

Let $a = T(e_1) \in \mathbb{R}$, $b = Te_2 \in \mathbb{R}$, $c = Te_3 \in \mathbb{R}$.

Now observe that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T(xe_1 + ye_2 + ze_3)$$
$$= xTe_1 + yTe_2 + zTe_3 = ax + by + cz.$$

c) $\text{null}\, T = \{ T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0 \mid x, y, z \in \mathbb{R} \}$

$= \{ ax + by + cz = 0 \mid x, y, z \in \mathbb{R} \}$

So null T is the set of pts which solve

$$ax + by + cz = 0$$

$\Rightarrow$ this is the eq for a plane.

d) Let $\vec{n}$ be the normal vector to the plane.

First, as $\vec{n}$ is $\perp$ to our plane $\vec{n} \notin \text{null}\, T$. so

$$r = T(\vec{n}) \neq 0.$$

Now if $s \in \mathbb{R}$, then $T(\frac{s}{r} \cdot \vec{n}) = \frac{s}{r} T(\vec{n}) = \frac{s}{r} \cdot r = s.$

so $T : \text{span}(\vec{n}) \to \mathbb{R}$ is surjective.

To see this mapping is injective assume

$$T(\alpha \vec{n}) = T(\beta \vec{n})$$

then $\alpha T(\vec{n}) = \beta T(\vec{n}) \Rightarrow \alpha = \beta$ (as $T(\vec{n}) \neq 0$)

$\therefore T : \text{span}(\vec{n}) \to \mathbb{R}$ is also injective.

3) a- Let $v, u \in V$. so $\exists$ scalars $a_i, b_i \in \mathbb{F}$ so that

$v = a_1 v_1 + \cdots + a_n v_n$ $\quad u = b_1 v_1 + \cdots + b_n v_n$ now

$$T(u+v) = T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n)$$

$$= (a_1+b_1)x_1 + \cdots + (a_n+b_n)x_n$$

$$= (a_1 x_1 + \cdots + a_n x_n) + (b_1 x_1 + \cdots + b_n x_n)$$

$$= T(u) + T(v).$$

A similiar calc shows $T(sv) = sTv$ $\quad \forall s \in \mathbb{F}$.

$\therefore T \in \mathcal{L}(V, W)$.

b- ($\Leftarrow$) By corollary 3.5, we know that
if T is surjective, then $\dim W \leq \dim V$

($\Rightarrow$) Now assume $\dim W \leq \dim V$. Let
$\{v_1 \ldots v_n\}$ + $\{w_1 \ldots w_m\}$
be bases for V + W, resp, where $m \leq n$.
By part a) define $T \in L(V, W)$ $\ni$.
$$Tv_i = w_i \quad (i \leq m) \quad + \quad Tv_i = 0_W \quad (i > m).$$
It can now be checked that T is surjective.

5) As $\text{null}\, T \leq V$ + $\text{ran}\, T \leq V$ are subspaces
then $\text{null}\, T + \text{ran}\, T$ is a subspace of V.
As their intersection is trivial, then
$$\text{null}\, T \oplus \text{ran}\, T \leq V.$$
By rank-nullity
$$\dim V = \dim(\text{null}\, T) + \dim(\text{ran}\, T) = \dim(\text{null} \oplus \text{ran})$$
$$\therefore \quad V = \text{null}\, T \oplus \text{ran}\, T.$$