solutions: HW # 3

1) ($\Rightarrow$) Assume $\mathcal{U}$ is a proper subspace of V. Let B be a basis for $\mathcal{U}$. Then

$$\text{span}(B) = \mathcal{U} \subsetneq V.$$

As B is indep in V but not a spanning set for V, Thm 2.10 implies that

$$\dim(\mathcal{U}) = |B| < \dim(V)$$

($\Leftarrow$) Consider the contrapositive. If $\mathcal{U} = V$, then $\dim(\mathcal{U}) = \dim(V)$. As this is obvious we are done.

2) No. To see this, assume for a contradiction that

$$\text{span}(S) = \mathcal{P}_{\leq n}.$$

By the Basis Reduction thm, $\exists$ some $B \subseteq S$ $\ni$ B is a basis for $\mathcal{P}_{\leq n}$. As $|S| = n$, then

$$n+1 = \dim(\mathcal{P}_{\leq n}) = |B| \leq |S| = n \quad \Rightarrow\Leftarrow$$

$\therefore$ $\text{span}(S) \neq \mathcal{P}_{\leq n}$.

3) a- First observe that $0_V \in \mathcal{U}$ and $0_V \in W$ so $0_V = 0_V + 0_V \in \mathcal{U}+W$. Next, if $u+w, u'+w' \in \mathcal{U}+W$ then

$$(u+w)+(u'+w') = \underbrace{(u+u')}_{\in \mathcal{U}} + \underbrace{(w+w')}_{\in W} \in \mathcal{U}+W.$$

$\Rightarrow$ so closed under addition. Lastly, if $s \in \mathbb{F}$, then

$$s(u+w) = \underbrace{su}_{\in \mathcal{U}} + \underbrace{sw}_{\in W} \in \mathcal{U}+W.$$

$\Rightarrow$ so closed under scalar mult. $\therefore$ subspace.

b- Let $V = \mathbb{R}^3$ and define

$$\mathcal{U} = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

and

$$W = \operatorname{span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right).$$

Observe $\mathcal{U} + W = \mathbb{R}^3$. Let $v = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$.

Consider $u = \begin{bmatrix}a\\b\\0\end{bmatrix}$, $u' = \begin{bmatrix}a\\0\\0\end{bmatrix} \in \mathcal{U}$ and

$$w = \begin{bmatrix}0\\0\\c\end{bmatrix}, \quad w' = \begin{bmatrix}0\\b\\c\end{bmatrix} \in W.$$

Then

$$u + w = v = u' + w'.$$

are ?.

c- Assume $u, u' \in \mathcal{U}$ and $w, w' \in W$

$$u + w = v = u' + w'.$$

To see this, we must show $u = u'$, $w = w'$. Now note that $u + w = u' + w'$.

$$u - u' = w - w' \quad (*)$$

By $(*)$ $u - u', w - w' \in \mathcal{U} \cap W = \{0_V\}$.

$\therefore$ $u = u'$ and $w = w'$ as needed.

d- First let
$$B = \{v_1 \cdots v_k, u_1 \cdots u_n, w_1 \cdots w_m\}.$$

To show B is a basis for $\mathcal{U}+W$, first we show $\text{span}(B) = \mathcal{U}+W$. Note:
$$\mathcal{U} \cup W \subseteq \text{span}(B).$$

so
$$\mathcal{U}+W = \text{span}(\mathcal{U} \cup W) \subseteq \text{span}(B) \subseteq \mathcal{U}+W$$

$\therefore$ $\text{span}(B) = \mathcal{U}+W.$

Next we must show B is indep. To this end, let a_i, b_i + c_i be scalars $\ni$.
$$0 = a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_nu_n + c_1w_1 + \cdots + c_mw_m$$

so,
$$-\underbrace{(c_1w_1 + \cdots + c_mw_m)}_{\in W} = \underbrace{a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_nu_n}_{\in \mathcal{U}}$$

This means $c_1w_1 + \cdots + c_mw_m \in W \cap \mathcal{U}$. As $\{v_1 \cdots v_k\}$ is a basis for this space, then
$$c_1w_1 + \cdots + c_mw_m = d_1v_1 + \cdots + d_kv_k$$
for some scalars d_i. As $\{w_1 \cdots w_m, v_1, \cdots v_k\}$ is indep then the c_i's + d_i's $= 0$. This means * becomes
$$0 = a_1v_1 + \cdots + a_kv_k + b_1u_1 + \cdots + b_nu_n$$
As $\{v_1, \cdots, v_k, u_1, \cdots, u_n\}$ is also indep, then a_i's $= b_i$'s $= 0$.

$\therefore$ The set B is indep. and B is thus a basis for $\mathcal{U}+W$.

If $U \cap W = \{0_V\}$, then

$$\dim(U \oplus W) = \dim(U) + \dim(V).$$

Linear Transformations

1) To prove $\text{null}\,T$ is a subspace, let $u, v \in \text{null}\,T$. Then

$$T(u+v) = Tu + Tv = 0_W + 0_W = 0_W$$

so $u+v \in \text{null}\,T$. (closed under addition) If $a \in \mathbb{F}$, then

$$T(au) = aT(u) = a \cdot 0_W = 0_W$$

so $au \in \text{null}\,T$. (closed under mult.) Lastly, by Lemma 3.1, $T(0_V) = 0_W$, so $0_V \in \text{Null}\,T$.

The proof for $\text{ran}\,T$ is similiar.

2) a) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x-y, z)$.

then $\text{null}\,T = \{(x,y,0) \mid x = y\}$. As this is a line in $\mathbb{R}^3$ it has dim 1. so T is <u>not</u> injective.

We claim $\text{ran}\,T = \mathbb{R}^2$. To see this, let $(a,b) \in \mathbb{R}$

then

$$T((a,0,b)) = (a,b)$$

so T is surjective. $\dim \text{ran}\,T = \dim \mathbb{R}^2 = 2$.

b) Consider $T: \mathcal{P}_{\leq 2} \to \mathcal{P}_{\leq 3}$ given by

$$Tf = xf + f'.$$

Let $f = a + bx + cx^2$. If $Tf = 0$, then

$$0 = x(a+bx+cx^2) + b + 2cx$$

or

$$0 = b + (a+2c)x + bx^2 + cx^3$$

so $c = b = 0$ and $a + 2c = 0$.

$\Rightarrow a = b = c = 0$.

It now follows that
$$\text{null}\, T = \{0\}$$
and T is injective. To determine the range of T consider the basis $\{1, x, x^2\}$ for $\mathcal{P}_{\leq 2}$.

Now
$$\begin{aligned}
\text{ran}\, T &= \{Tp \mid p \in \mathcal{P}_{\leq 2}\} \\
&= \{T(a \cdot 1 + bx + cx^2) \mid a, b, c \in \mathbb{R}\} \\
&= \{aT(1) + bT(x) + cT(x^2) \mid a, b, c \in \mathbb{R}\} \\
&= \text{span}\,(T(1), T(x), T(x^2)) \\
&= \text{span}\,(x, x^2+1, x^3+2x).
\end{aligned}$$

As x, x^2+1, and x^3+2x are indep (check!) then
$$\dim(\text{ran}\, T) = 3.$$
As $\dim(\mathcal{P}_{\leq 3}) = 4$, T is <u>not</u> surjective.

3)

a- As T is injective, then $\text{null}\, T = \{0_V\}$. ~~As~~ Let $v_1, \dots, v_m \in S$. To show $T(S)$ is an indep set assume $a_1, \dots, a_m \in \mathbb{F}$ such that
$$a_1 T(v_1) + \cdots + a_m T(v_m) = 0_W. \quad \circledast$$
we must show $a_1 = \cdots = a_m = 0$. The linearity of T implies that
$$T(a_1 v_1 + \cdots + a_m v_m) = 0_W.$$
so $a_1 v_1 + \cdots + a_m v_m \in \text{null}\, T = \{0_V\}$. This means
$$a_1 v_1 + \cdots + a_m v_m = 0_V.$$
As S is indep, we conclude that $a_1 = \cdots = a_m = 0$

b- Now assume T is surjective. As $\operatorname{span}(S) = V$ then

$$\operatorname{span}(T(S)) = \{a_1 Tv_1 + \cdots + a_k Tv_k \mid v_1 \dots v_k \in S,\ a_1 \dots a_k \in \mathbb{F}\}$$

$$= \{T(a_1v_1 + \cdots + a_k v_k) \mid v_i \in S,\ a_i \in \mathbb{F}\}$$

$$= \{Tv \mid v \in \operatorname{span}(S)\}$$

$$= \{Tv \mid v \in V\} = \operatorname{ran} T \underset{\substack{\uparrow \\ \text{as } T \text{ is} \\ \text{surjective.}}}{=} W$$

c- As T is injective, then $T(\mathcal{B})$ is indep. As T is surjective, then $T(\mathcal{B})$ spans W.

$\therefore$ $T(\mathcal{B})$ is a basis for W.

b- Now assume T is surjective, so $\operatorname{ran} T = W$.

As S spans V, then

$$\operatorname{span}(S(T)) = \{$$