Solutions: HW 2

1) To show that span S is a subspace of V, we break it into two cases: If $S = \emptyset$, then span $\emptyset = \{0_V\}$ by definition, which is the trivial subspace. Now assume $S \neq \emptyset$. Let $v \in S$.

then $0_V = 0 \cdot v \in \text{span}(S)$.

To check the closure axioms, let $u, w \in$ span S. This means $\exists$ $v_1, \ldots, v_m \in S$ and $a_1, \ldots, a_m \in \mathbb{F}$, $b_1, \ldots, b_m \in \mathbb{F}$ so that

$$u = a_1 v_1 + \cdots + a_m v_m \quad \text{and} \quad w = b_1 v_1 + \cdots + b_m v_m$$

Now

$$u + w = (a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_m v_m)$$
$$= (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m \in \text{span}(S)$$

$\Rightarrow$ so span(S) is closed under addition.

If $s \in \mathbb{F}$, is any scalar, then

$$s \cdot u = s(a_1 v_1 + \cdots + a_m v_m) = s a_1 v_1 + \cdots + s a_m v_m \in \text{span}(S)$$

$\Rightarrow$ so span(S) is closed under scalar mult.

$\therefore$ span(S) is a subspace of V.

2) To prove that $T = \{v_1 - v_2, \ldots, v_{n-1} - v_n, v_n\}$ spans V it will suffice to show that

$v_i \in \text{span}(T)$ $\forall$ values of i.

(why?) First, observe that $v_n \in T \subseteq \text{span}(T)$.
By induction, we may now assume $v_k, \ldots, v_n \in \text{span}(T)$ for some $k \leq n$.

As $v_{k-1} - v_k \in T \subseteq \text{span}(T)$, we must have

$$v_{k-1} = (v_{k-1} - v_k) + v_k \in \text{span}(T)$$

which completes our induction.

$\therefore$ $v_1, \ldots, v_n \in \text{span}(T)$ as needed.

3) a- ($\Rightarrow$) Assume $\{u, v\}$ is dep set. Then $\exists$ scalars $a, b \in \mathbb{F}$, not both zero $\ni$. $au + bv = 0$

wlog, assume $a \neq 0$. Then

$$u = -\frac{b}{a} v,$$

i.e., u is a multiple of v.

($\Leftarrow$) Now assume u is a multiple of v. So $u = cv$, for some $c \in \mathbb{F}$, and we have

$$1 \cdot u - cv = 0.$$

As u + v are distinct (this is needed, why?) this implies $\{u, v\}$ is dependent.

b - consider the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

4) ($\Rightarrow$) Assume $\mathcal{B}$ is a basis for V. Now fix $u \in V$. As $\mathcal{B}$ spans all of V, $\exists$ $v_1, \ldots, v_m \in \mathcal{B}$ and scalars $a_1, \ldots, a_m \in \mathbb{F}$, $\ni$.

$$u = a_1 v_1 + \cdots + a_m v_m.$$

To show this is unique, assume

$$u = b_1 v_1 + \cdots + b_m v_m$$

for some other choice of scalars b_i. (Note we may assume the vectors $v_1, \ldots, v_m$ are the same in both cases by allowing the respective coef to be zero.)

Now

$$0_V = u - u = (a_1 - b_1) v_1 + \cdots + (a_m - b_m) v_m.$$

As $\mathcal{B}$ is also linearly indep, we must have $a_i - b_i = 0$ or $a_i = b_i$. Therefore u has a <u>unique</u> linear combination in terms of the basis $\mathcal{B}$.

($\Leftarrow$) Now assume every vector $u \in V$ can be written uniquely as

$$u = a_1 v_1 + \cdots + a_m v_m$$

for some $a_i \in \mathbb{F}$ & $v_i \in \mathcal{B}$. This means that

$$V = \operatorname{span}(\mathcal{B}).$$

To see that $\mathcal{B}$ is linearly indep, note if

$$0_V = c_1 w_1 + \cdots + c_n w_n$$

for some choice of $c_i \in \mathbb{F}$ and $w_i \in \mathcal{B}$, then $c_1 = \cdots = c_n = 0$. So the uniqueness implies that $\mathcal{B}$ is linearly indep.

$\therefore$ $\mathcal{B}$ is a basis

5) Let $p_0, \dots, p_n \in \mathcal{P}_{\le n}$, such that $p_i(0) = 0$.
This means that each p_i can be factored as

$$p_i = X \cdot q_i$$

since 0 is a root of each p_i. Now observe that the p_i are linearly dep iff the q_i are.
But deg $q_i \le n-1$, so we have

$$q_0, \dots, q_n \in \mathcal{P}_{\le n-1}.$$

As dim $\mathcal{P}_{\le n-1}$ is n, any set of $n+1$ vectors must be linearly dependent (Thm 2.10). Therefore $q_0, \dots, q_n$ is linearly dep, and by what we said above, so are the original $p_0, \dots, p_n$.

6) Let $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ be bases for U and W, respectively. If $U \cap W = \{0_V\}$, we will show that together, the $2n$ (distinct) vectors

$$S = \{u_1, \dots, u_n\} \cup \{w_1, \dots, w_n\}$$

are linearly indep. This a contradiction since any set of indep vectors in V has size at most $2n-1$.
To this end let a_i & b_i be s.t.

$$0_V = a_1 u_1 + \dots + a_n u_n + b_1 w_1 + \dots + b_n w_n$$

This means

$$v = \underbrace{b_1 w_1 + \dots + b_n w_n}_{\in W} = -\underbrace{(a_1 u_1 + \dots + a_n u_n)}_{\in U}$$

so $v \in W \cap U = \{0_V\}$.

Therefore

$$0_V = b_1 w_1 + \cdots + b_n w_n.$$

As the w_i are indep we get that $b_1 = \cdots = b_n = 0$.

Likewise (as the u_i are indep) we see that

$a_1 = \cdots = a_n = 0$.

$\therefore$ The set S is linearly indep.

7) Let V have dim n and consider $S \subseteq V$ $\ni$.

$$\text{span}(S) = V.$$

To prove this, let $\mathcal{B}$ be a linearly indep subset of S w/ largest size possible. (That is, out of all the indep subsets of S, let $\mathcal{B}$ be one w/ maximal cardinality.) If $\text{span}(\mathcal{B}) = V$, then we are done, as $\mathcal{B}$ is therefore a basis.

If not, then, as $\text{span}(S) = V$, we must have some

$$u \in S \setminus \text{span}(\mathcal{B}).$$

Then by the linear indep lemma we have that

$$\mathcal{B} \cup \{u\}$$

is an indep set, which contradicts our choice of $\mathcal{B}$.

<u>Note:</u> where did we use that V is finite-dimensional?

$\Rightarrow$ It was needed somewhere!