Solutions: HW #1

1) We first note that $M_{r,c}$ under matrix addition and scalar mult is clearly associative, commutative, and the dist law holds. Further $1 \in \mathbb{F}$ is a mult identity, as $1 \cdot A = A$, $\forall A \in M_{r,c}$. As the zero matrix is the additive identity, and $-1 \cdot A$ is an additive inverse. So $M_{r,c}$ is a vector space over $\mathbb{F}$.

b. To see that the set of diagonal matrices $D \subseteq M_{n,n}$ is a subspace we must check 3 properties.

First, note that the zero vector in $M_{r,c}$ is a diagonal matrix, we see the zero vector is in D.

Now let $A, B \in D$, so that

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix}$$

Next, note that D is closed under addition as

$$A + B = \begin{pmatrix} a_1+b_1 & & 0 \\ & \ddots & \\ 0 & & a_n+b_n \end{pmatrix} \in D.$$

Lastly, D is closed under scalar mult as

$$s \cdot A = s \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \begin{pmatrix} sa_1 & & 0 \\ & \ddots & \\ 0 & & sa_n \end{pmatrix} \in D$$

for $s \in \mathbb{F}$.

2) yes, To see this, observe that the zero vector in $\mathcal{C}(\mathbb{R})$ is the zero fct $f(x) = 0 \ \forall x \in \mathbb{R}$. so the zero vector is in $\mathcal{U}$. To see that $\mathcal{U}$ is closed under addition let $g, h \in \mathcal{U}$. Now

$$(g+h)(1) = g(1) + h(1) = 0 + 0 = 0$$

So $g+h \in \mathcal{U}$. and $\mathcal{U}$ is closed under addition. Likewise, for any scalar $s \in \mathbb{R}$,

$$(s \cdot g)(1) = s g(1) = s \cdot 0 = 0$$

So $s \cdot g \in \mathcal{U}$, and we see that $\mathcal{U}$ is closed under scalar mult.

$\therefore$ $\mathcal{U}$ is a subspace.

b - NO. To prove this it will suffice to show $\mathcal{U}'$ is not closed under addition. This is indeed true as

$$x, x^2 \in \mathcal{U}', \text{ yet } x + x^2 \notin \mathcal{U}'.$$

3) consider the set $\mathbb{Z}^2 = \{ (a,b) \mid a, b \in \mathbb{Z} \}$.

4) As $0_V \in \mathcal{U}$ and $0_V \in W$, $0_V \in \mathcal{U} \cap W$. Now let $u, v \in \mathcal{U} \cap W$. This means $u, v \in \mathcal{U}$ and $u, v \in W$. As $\mathcal{U}$ and W are vector spaces we know that

$$u + v \in W \quad \text{and} \quad u + v \in \mathcal{U}.$$

so $u + v \in W \cap \mathcal{U}$.- For the same reason,

$$s \cdot u \in W \quad \text{and} \quad s \cdot u \in \mathcal{U} \qquad (\forall s \in \mathbb{F})$$

so $s \cdot u \in W \cap \mathcal{U}$. Therefore $W \cap \mathcal{U}$ is a subspace.

b- ($\Leftarrow$) wlog assume $\mathcal{U} \subseteq W$. Then $\mathcal{U} \cup W = W$. As W is a subspace we see that $\mathcal{U} \cup W$ is too.

($\Rightarrow$) If $W = \mathcal{U}$, we are done. Otherwise, wlog let us assume $w \in W \setminus \mathcal{U}$. As $\mathcal{U} \cup W$ is a subspace, then $\forall\, u \in \mathcal{U}$, $u + w \in \mathcal{U} \cup W$.

This means that either, $u + w \in \mathcal{U}$ or $u + w \in W$.

In fact, we see that $u + w \notin \mathcal{U}$. If it were then $u + w = u' \in \mathcal{U}$ for some $u' \in \mathcal{U}$.

But then

$$w = u' - u \in \mathcal{U}$$

which contradicts the fact that $w \in W \setminus \mathcal{U}$.

so $u + w \in W$. Now $u + w = w' \in W$ for some $w' \in W$, and

$$u = w' - w \in W.$$

As this is true for all $u \in \mathcal{U}$, then $\mathcal{U} \subseteq W$.

5) To show $0 \cdot v = 0_v$ for all $v \in V$, note that

$$0 \cdot v = (0 + 0)v = 0v + 0v$$

Adding $-0 \cdot v$ to both sides

$$0 \cdot v + -0 \cdot v = 0 \cdot v + 0 \cdot v + -0 \cdot v$$

which reduces to $0_v = 0 \cdot v$. Likewise

$$a \cdot 0_v = a \cdot (0_v + 0_v) = a 0_v + a 0_v.$$

Adding $-a \cdot 0_v$ to both sides, and reducing yields

$$0_v = a \cdot 0_v.$$

b) Assume $a \cdot v = 0_V$. If $a \neq 0$, then

$$v = 1 \cdot v = \frac{1}{a} \cdot a \cdot v = \frac{1}{a} 0_V = 0_V.$$

c) Assume, 1 and $1' \in \mathbb{F}$ are both mult id.
As V is nontrivial, $\exists\ w \in V$ w/ $w \neq 0_V$.
Now

$$0_V = w - w = 1 \cdot w - 1' \cdot w = (1-1')w$$

As $w \neq 0_V$, then part b) implies $1-1' = 0$

$\therefore\ 1 = 1'$ as claimed.