Math 350
Abstract Linear Algebra
Final Exam Practice

For the first two problems, assume V is an n-dimensional complex vector space.

1. Let $N \in \mathcal{L}(V)$ be a nilpotent operator.
   (a) Show that 0 is the only eigenvalue for N.
   (b) Assuming V has Jordan basis $\mathcal{J} = \{v, Nv, \ldots, N^k v\}$, with respect to N, find the dimension of the eigenspace corresponding to 0 for N. What is the dimension of the generalized eigenspace corresponding to 0 for N?

2. Assume $T \in \mathcal{L}(V)$ has characteristic polynomial $\rho_T(x) = (x-2)^2(x-3)$.
   (a) Must T be diagonalizable?
   (b) What if $f(T) = 0$, where $f(x) = (x-2)(x-3)$. (Hint: Consider Jordan Form.)

For the remaining problems, assume V is an n-dimensional $\mathbb{F}$-inner product space.

3. Let U be a subspace of V and consider the projection operator $P = P_U$.
   (a) Prove that P is a linear map.
   (b) Prove that $\|Px\| \leq \|x\|$.
   (c) Let $T \in \mathcal{L}(V)$ be another operator. Show that U is T-invariant if and only if $PTP = TP$.

4. Recall that an operator $U \in \mathcal{L}(V)$ is said to be **unitary** provided that

$$< Uv, Uw >=< v, w >, \quad \text{for all } v, w \in V.$$

   Prove that any two eigenvectors for U, which correspond to distinct eigenvalues, are orthogonal.

(over)

Optional problem

1. Let V be an $\mathbb{F}$-inner product space and let $P \in \mathcal{L}(V)$ be such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|,$$

for all $v \in V$. Prove that P is an orthogonal projection. (Hint: First prove that $< u, v >= 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.)