Math 350
Abstract Linear Algebra

Practice set #2

You may assume that V and W are finite-dimensional $\mathbb{F}$-vector spaces throughout.

1. Let $T \in \mathcal{L}(V, W)$ be an isomorphism. Prove that if U is a subspace of V, then $\dim T(U) = \dim U$.

2. Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (2x + y, x - 3y)$ and consider the two bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\},$$

   for $\mathbb{R}^2$.

   (a) Find $[T]_{\mathcal{B}}$.
   (b) Find $[T]_{\mathcal{C}}$ using the Change of Basis theorem.

3. Let U and W be T-invariant subspaces of V.

   (a) Prove that $U \cap W$ is also T-invariant.
   (b) If U is also invariant under every operator on V, must $U = V$ or $U = \{0_V\}$?

4. Let $S, T \in \mathcal{L}(V)$. Assume $ST = TS$ and let (v, λ) be an eigenpair for T. Provided $Sv \neq 0_V$, prove that (Sv, λ) is also an eigenpair for T.

5. Fix a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V and let T be an operator on V so that

$$Tv_i = \begin{cases} v_{i+1} & \text{if } i < n \\ v_1 & \text{if } i = n \end{cases}.$$

   Find an eigenpair for T. (Hint: This is possible without messy calculations.)

6. (Proof of Theorem 4.6) Let $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if V has a basis consisting entirely of eigenvectors for T. Conclude that if T has $\dim V$ distinct eigenvalues, then it is diagonalizable.

Optional problem

Let $T \in \mathcal{L}(V)$. Prove there must exist some $k > 0$ such that

$$V = \operatorname{ran} T^k \oplus \operatorname{null} T^k.$$