Math 350
Abstract Linear Algebra
Practice Set #1

Let V and W be $\mathbb{F}$-vector spaces unless otherwise stated.

1. Is $\mathbb{R}^2$ an $\mathbb{R}$-vector space where addition and scalar multiplication are given by

$$(a, b) + (c, d) = (a + c, bd) \quad \text{and} \quad s(a, b) = (sa, sb),$$

where $s \in \mathbb{R}$?

2. Let V be an n-dimensional space over $\mathbb{C}$. Prove that V, viewed as a vector space over $\mathbb{R}$ has dimension $2n$.

3. Let $\mathcal{M}_n$ be the $\mathbb{R}$-vector space of all $n \times n$ matrices with entries in $\mathbb{R}$. Define the map $\mathrm{Tr} : \mathcal{M}_n \to \mathbb{R}$ by

$$\mathrm{Tr}\left(\begin{bmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ * & & & a_n \end{bmatrix}\right) = a_1 + a_2 + \cdots + a_n$$

   (a) Prove $\mathrm{Tr} \in \mathcal{L}(\mathcal{M}_n, \mathbb{R})$.
   (b) Describe $\operatorname{null} \mathrm{Tr}$ and determine its dimension.

4. Let

$$U = \{p \in \mathcal{P}_{\leq n} \mid p(-x) = -p(x) \text{ for all } x \in \mathbb{R}\}$$

be the set of all odd polynomials and

$$W = \{p \in \mathcal{P}_{\leq n} \mid p(-x) = p(x) \text{ for all } x \in \mathbb{R}\},$$

be the set of all even polynomials.

   (a) Prove that U and W are subspaces of $\mathcal{P}_{\leq n}$.
   (b) Determine $\dim U$ and $\dim W$.
   (c) Conclude that $\mathcal{P}_{\leq n} = U \oplus W$.

5. Suppose V is finite-dimensional and fix $T \in \mathcal{L}(V, W)$. Prove that there exists some subspace U of V such that $U \cap \operatorname{null} T = \{0_V\}$ and $\operatorname{ran} T = \{Tu \mid u \in U\}$. Conclude that

$$V = \operatorname{null} T \oplus U.$$