Math 350
Abstract Linear Algebra
Homework Set #9

Inner Product Spaces

You may assume that $(V, <, >)$ is a finite-dimensional $\mathbb{F}$-inner product space.

1. Assuming V is a real inner product space, verify that
$$< u, v >= \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

2. Suppose that $\{e_1, \ldots, e_k\}$ is an orthonormal set in V. Show that
$$\|v\|^2 = | < v, e_1 > |^2 + \cdots + | < v, e_k > |^2,$$
if and only if $v \in \operatorname{span}(e_1, \ldots, e_k)$.

3. In this problem, assume $\mathbb{F} = \mathbb{C}$. We say an operator $U \in \mathcal{L}(V)$ is **unitary** provided that
$$< Uv, Uw >=< v, w >, \quad \text{for all } v, w \in V.$$
   (a) Prove that U preserves length, i.e.,
$$\|v\| = \|Uv\|, \quad \text{for all } v \in V.$$
   (b) Prove the eigenvalues of a unitary operator lie on the unit circle in $\mathbb{C}$.
   (c) Using (b), prove that unitary operators are invertible.
   (d) In this part let $V = \mathbb{C}^n$ and $<, >$ be the usual dot product. Assuming $A : \mathbb{C}^n \to \mathbb{C}^n$ is a unitary (matrix) operator, prove the columns of A form an orthonormal basis for $\mathbb{C}^n$. Is the converse true? If so, prove it!

4. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that if $\operatorname{null} P \subseteq (\operatorname{ran} P)^{\perp}$, then P is an orthogonal projection. (Hint: First show that $V = \operatorname{ran} P \oplus \operatorname{null} P$.)