Math 350
Abstract Linear Algebra
Homework Set #8

Jordan Form

1. Let S and T be operators on V. Prove that if ST is nilpotent then TS must also be nilpotent. (Note: You cannot assume $ST = TS$.)

2. Let N is a nilpotent operator on V. Fix $k \geq 0$ and define

$$S = \{v, Nv, N^2v, \ldots, N^kv\}.$$

   Assuming $N^kv \neq 0_V$, show that S is linearly independent.

3. Let T be an complex operator on a 4-dimensional space whose only eigenvalue is 2.
   (a) Find all the possible Jordan forms for T (up to rearrangement of the Jordan blocks).
   (b) For matrix J found in part (a), determine the minimal k so that $(J-2)^k = 0$.

Inner Product Spaces

1. Let $\mathcal{C}([0,1])$ be the real vector space of continuous function on the interval $[0,1]$. In class we claimed that

$$< \cdot, \cdot >: \ \mathcal{C}([0,1]) \times \mathcal{C}([0,1]) \to \mathbb{R},$$

   given by

$$< f, g > := \int_0^1 f(x)g(x) \ dx,$$

   is an inner product.
   (a) Verify this claim.
   (b) Find $\|x^n\|$.
   (c) Show that $\cos(\pi x)$ is orthogonal to $\sin(\pi x)$ under this inner product.

2. We say an $n\times n$ complex valued matrix is **positive-definite** provided it satisfies the following two conditions. First, we require that

$$0 < Ax \cdot x,$$

for all $\vec{0} \neq x \in \mathbb{C}^n$ where $\cdot$ is the standard dot product. Additionally, we require $A^* = A$. (The symbol $*$ means the conjugate transpose; so A^* is obtained by taking the transpose and then conjugating each complex-valued entry.)

Fix an $n \times n$ positive-definite matrix A and let V be an n-dimensional complex vector space with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Define a function

$$< \cdot, \cdot >_A \colon\ \mathcal{B} \times \mathcal{B} \to \mathbb{C},$$

by $< v_i, v_j >_A = A_{i,j}$, where $A_{i,j}$ is the (i,j)th entry of A. Now extend this function to all of $V \times V$, by defining

$$< v, w >_A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{b_j} < v_i, v_j >_A .$$

where $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$.

(a) First, convince yourself that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{b_j} A_{i,j} = Ax \cdot y,$$

where $y = \begin{bmatrix} \overline{a_1} \\ \vdots \\ \overline{a_n} \end{bmatrix}$ and $x = \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix}$.

(b) Verify that $< \cdot, \cdot >_A$ is an inner product on V.

(c) Prove that if A is the identity matrix, then $\mathcal{B}$ is an orthonormal basis under $< \cdot, \cdot >_A$. (Note: This means that every basis for V is orthonormal under **some** inner product.)

(d) Show that if $< \cdot, \cdot >$ is an arbitrary inner product on V, then there exists some positive-definite matrix A so that

$$< \cdot, \cdot > = < \cdot, \cdot >_A .$$