Math 350
Abstract Linear Algebra
Homework Set #7

Generalized Eigenspaces

Throughout you may assume that V is a complex vector space of degree n and that $T \in \mathcal{L}(V)$.

1. Prove Lemma 4.10 in the notes.

2. Prove that $T^n = 0$ if and only if 0 is the only eigenvalue for T.

3. (a) Let $\rho_T(x)$ be the characteristic polynomial of T. Prove that T is invertible if and only if $\rho_T(0) \neq 0$.
   (b) Using part (a), conclude that if T is invertible, then there exists a polynomial $p(x)$ such that $T^{-1} = p(T)$.

4. Let $\lambda_1, \ldots, \lambda_m$ be T's distinct eigenvalues. The dimension of $\text{null}(T - \lambda_i)$ is commonly referred to as the **geometric multiplicity** of λ_i. Likewise, the dimension of $\mathcal{G}_i = \text{null}(T - \lambda_i)^n$ is often called the **algebraic multiplicity** of λ_i.
   (a) Prove for each λ_i that
   $$\text{null}(T - \lambda_i) \subset \mathcal{G}_i\,.$$
   Conclude that the algebraic multiplicity of λ_i is at least as large as its geometric multiplicity.
   (b) From part (a), show that $\mathcal{G}_i$ has a basis consisting entirely of eigenvectors if and only if $\mathcal{G}_i = \text{null}(T - \lambda_i)$.
   (c) Conclude that T is diagonalizable if and only if the algebraic multiplicity of λ_i equals its geometric multiplicity for all eigenvalues λ_i. (Hint: Use Theorem 4.15.)

Optional problem

The goal of this problem is to prove that any monic degree n polynomial with complex coefficients is the characteristic polynomial of some operator on V. Set $a_0, \dots, a_{n-1} \in \mathbb{C}$ and consider the operator $A : \mathbb{C}^n \to \mathbb{C}^n$, where

$$A = \begin{bmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & \ddots & -a_2 \\ & & \ddots & \vdots \\ & & & -a_{n-1} \end{bmatrix}.$$

Show that $\rho_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.