Math 350
Abstract Linear Algebra
Homework Set #6

Eigenvectors & Eigenvalues

1. Let $T \in \mathcal{L}(V)$.
   (a) Prove T is invertible if and only if 0 is *not* an eigenvalue for T.
   (b) Now assume T is invertible. Prove λ is an eigenvalue for T if and only if λ^{-1} is an eigenvalue for T^{-1}.

2. Assume $\dim(\operatorname{Ran} T) = k$. Prove that T has at most $k+1$ distinct eigenvalues.

3. (a) Find all the eigenvalues/vectors for the operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by:
$$T(x, y, z) = (2y, 0, 5z).$$
   (b) Given an example of an operator on $\mathbb{R}^4$ that has no (real) eigenvalues.

4. Let S and T be operators on V. Prove that ST and TS have the same eigenvalues.

5. Let $T \in \mathcal{L}(V)$ such that every subspace with dimension $\dim V - 1$ is T-invariant.
   (a) Prove that every (nonzero) $v \in V$ must be an eigenvector for T.
   (b) Prove that T has exactly one (distinct) eigenvalue λ.
   (c) Conclude that $T = \lambda I_V$.