Math 350
Abstract Linear Algebra
Homework Set #5

Linear Maps & Matrices

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x \\ 2x - y \end{bmatrix}.$$

   Compute $[T]_{\mathcal{B}}^{\mathcal{C}}$ where the bases $\mathcal{B}$ and $\mathcal{C}$ are as follows.

   (a) $\mathcal{B}$ and $\mathcal{C}$ are the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

   (b) $\mathcal{B}$ is the standard basis for $\mathbb{R}^2$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$.

   (c) Compute $T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$ directly and by using Theorem 3.11. Do your answers agree?

2. Define the linear maps $T : \mathbb{R}^4 \to \mathcal{P}_{\leq 3}$ by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = 2a + (a + b)x + (a + c)x^2 + (a + d)x^3$$

   and $S : \mathcal{P}_{\leq 3} \to \mathcal{P}_{\leq 2}$ by the derivative map $Sp = p'$. With $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ the standard bases on $\mathbb{R}^4$, $\mathcal{P}_{\leq 3}$ and $\mathcal{P}_{\leq 2}$ respectively, compute the following:

   (a) $[ST]_{\mathcal{B}}^{\mathcal{D}}$ directly.

   (b) $[S]_{\mathcal{C}}^{\mathcal{D}}$ and $[T]_{\mathcal{B}}^{\mathcal{C}}$.

   (c) $[ST]_{\mathcal{B}}^{\mathcal{D}}$ (again) using Theorem 3.12.

3. Let V be an n-dimensional vector space with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. In class we showed that all $V \cong \mathbb{R}^n$ since they both have n dimensions. Prove that the map $T : V \to \mathbb{R}^n$ given by $Tv = [v]_{\mathcal{B}}$ is an isomorphism between V and $\mathbb{R}^n$.

4. Let V be an n-dimensional space and let $T : V \to V$ be a linear map. Assume there exists a k-dimensional subspace U of V such that $T(U) \subset U$. Show there exists a basis $\mathcal{B}$ for V so that $n \times n$ matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ has the form

$$\left[\begin{array}{c|c} A & B \\ \hline O & D \end{array}\right],$$

where A, has size $k \times k$, O is the $(n-k) \times k$ zero matrix, and B and C are then matrices of the appropriate size.

Optional

1. Let V and W be vector spaces with the same dimension and let $T \in \mathcal{L}(V, W)$. Prove there exists bases $\mathcal{B}$ for V and $\mathcal{C}$ for W such that $[T]_{\mathcal{B}}^{\mathcal{C}}$ is a diagonal matrix.