Math 350
Abstract Linear Algebra
Homework Set #4

Rank-Nullity

1. Prove that if $T : \mathbb{R}^4 \to \mathbb{R}^2$ is linear and

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 2x_4 \text{ and } x_2 = 7x_3\},$$

then T is surjective.

2. Let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$ be a nonzero linear map.

   (a) Prove that $\dim(\operatorname{null} T) = 2$.

   (b) Prove there exists $a, b, c \in \mathbb{R}$ such that $T(x, y, z) = ax + by + cz$.

   (c) Show that $\operatorname{null} T$ is a plane in $\mathbb{R}^3$. Give the equation for this plane.

   (d) Let $\vec{n}$ be the normal vector to the plane you found in part c). Show that $T : \operatorname{span}(\vec{n}) \to \mathbb{R}$ is bijective.

3. For this problem let V and W be finite-dimensional vectors spaces and assume $\{v_1, \ldots, v_n\}$ is a basis for V.

   (a) Fix vectors $x_1, \ldots, x_n \in W$. Define a function

$$T : \{v_1, \ldots, v_n\} \to \{x_1, \ldots, x_n\}$$

   by $Tv_i = x_i$. Show that the "extended" function $T : V \to W$ given by

$$T(a_1 v_1 + \cdots a_n v_n) = a_1 T v_1 + \cdots + a_n T v_n = a_1 x_1 + \cdots a_n x_n$$

   is a linear transformation.

   (b) Let V and W be finite-dimensional vector spaces. Prove there exists a surjective linear map from V to W iff $\dim V \geq \dim W$. (Hint: Use part (a).)

4. Assume V is a finite-dimensional space and $T \in \mathcal{L}(V, V)$ such that

$$\operatorname{null} T \ \cap \ \operatorname{ran} T = \{0_V\}.$$

Prove that

$$V = \operatorname{null} T \oplus \operatorname{ran} T.$$