Math 350
Abstract Linear Algebra
Homework Set #3

Throughout let V, W be a finite-dimensional vector space over $\mathbb{F}$.

Dimension

1. Prove that U is a proper subspace of V iff $\dim U < \dim V$.
2. Assume you are given a set S of n polynomials in $\mathcal{P}_{\leq n}(\mathbb{R})$. Is it possible for S to span all of $\mathcal{P}_{\leq n}(\mathbb{R})$? Justify your answer.
3. Let U and W be subspaces of V. In general we denote the set

$$\{u + w \mid u \in U, w \in W\},$$

by $U + W$ and call it the **sum** of U and W. In the special case that $U \cap W = \{0_V\}$, we denote this set by $U \oplus W$ and call it the **direct sum** of U and W.

(a) Prove that $U + W$ is a subspace of V.

(b) Give an example of some $U, W \subsetneq V$ with the property that for every $v \in U + W$ there exist distinct $u, u' \in U$ and distinct $w, w' \in W$ such that

$$u + w = v = u' + w'.$$

(c) Now assume $U \cap W = \{0_V\}$. Prove that for every $v \in U \oplus W$, there is a *unique* choice of $u \in U$ and $w \in W$ such that

$$v = u + w.$$

(d) Let $\mathcal{B}_{U \cap W} = \{v_1, \ldots, v_k\}$ be a basis for $U \cap W$. By the Basis Extension Theorem, we know that there exists a basis $\mathcal{B}_U = \{v_1, \ldots, v_k, u_1, \ldots, u_n\}$ of U and a basis $\mathcal{B}_W = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}$ of W. Prove that

$$= \{v_1, \ldots, v_k, u_1, \ldots, u_n, w_1, \ldots, w_m\}$$

is a basis for $U + W$. Conclude that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

What does this say about the dimension of a direct sum?

Linear Transformations

1. Let $T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T$ is a subspace of V and that $\operatorname{range} T$ is a subspace of W.

2. For each of the following, prove the given function T is a linear map. Moreover, determine $\operatorname{range} T$ and $\operatorname{null} T$. Use this information to determine if T is injective/surjective.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = (x - y, z)$.

   (b) $T : \mathcal{P}_{\leq 2} \to \mathcal{P}_{\leq 3}$ given by $T(f)(x) = xf(x) + f'(x)$.

3. Let $T : V \to W$ be a linear map and $S \subset V$. Define

$$T(S) = \{Tv \mid v \in S\}.$$

   (a) Assume T is injective. Prove that if S is linearly independent, then $T(S)$ is an independent set in W.

   (b) Assume T is surjective. Prove that if S spans V, then $T(S)$ spans W.

   (c) If T is an isomorphism, conclude that if $\mathcal{B}$ is a basis for V, then $T(\mathcal{B})$ is a basis for W.