Math 350
Abstract Linear Algebra
Homework Set #1

Vector Spaces and Subspaces

---

1. (a) Define $M_{r,c}$ be the set of all matrices with r rows and c columns whose entries are in $\mathbb{R}$. Show that $M_{r,c}$ is a vector space over $\mathbb{R}$ under the usual operations of matrix addition and scalar multiplication.

   (b) Let D be the subset of the square matrices $M_{n,n}$ whose only nonzero entries lie on the diagonal. Show that D is a subspace of $M_{n,n}$.

2. (a) Recall that $\mathcal{C}(\mathbb{R})$ is the vector space over $\mathbb{R}$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define U to be the subset of all such function f such that $f(1) = 0$. Is U a subspace of $\mathcal{C}(\mathbb{R})$?

   (b) What if we consider the subset U' consisting of all continuous function such that $f(1) = 1$ instead?

3. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses but U is not a subspace of $\mathbb{R}^2$.

For the remainder, assume V is a vector space over $\mathbb{F}$ and $v \in V$ and $a \in \mathbb{F}$.

4. Assume U and W are subspaces of V.

   (a) Prove that $U \cap W$ is also a subspace of V.

   (b) Prove that $U \cup W$ is a subspace of V if and only if either $U \subset W$ or $W \subset U$.

5. (a) Prove Lemma 1.4 from class, i.e., show that

   $$0 \cdot v = 0_V \text{ for all } v \in V,$$

   and

   $$a \cdot 0_V = 0_V \text{ for all } a \in F.$$

   (b) Prove that if $av = 0_V$, then $a = 0$ or $v = 0$.

   (c) Assuming V is not the trivial vector space, prove that $1 \in \mathbb{F}$ is the unique scalar such that $1 \cdot v = v$ for all $v \in V$.