Mathematicians' perspectives on their pedagogical practice with respect to proof

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Abstract. In this paper, nine mathematicians were interviewed about their why and how they presented proofs in their advanced mathematics courses. Key findings include that (a) the participants in this study presented proofs not to convince students that theorems were true but for reasons such as conveying understanding and illustrating methods, (b) participants believed students did not appreciate the complex processes involved in reading a proof but often did not teach these processes to students, (c) the participants used superficial methods to assess students’ understanding of a proof, and (d) some participants questioned whether proof was the best way to convey mathematics to all of their students.

Key words: Mathematicians’ pedagogy; Proof; Proof presentation; Undergraduate mathematics education

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1. Introduction

In lamenting the quality of instruction in university mathematics courses, Davis and Hersh (1981) contended that “a typical lecture in advanced mathematics… consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation” (p. 151). Similarly, Dreyfus (1991) writes that the typical mathematics instructor teaches “almost exclusively the one very convenient and important aspect which has been described above, namely the polished formalism, which so often follows the sequence theorem-proof-application” (p. 27). Although lectures in advanced mathematics may be somewhat more varied than the descriptions above (e.g., Weber, 2004; Fukawa-Connelly, 2007), students do spend a significant amount of time reading the proofs that their professors present to them in lectures.

If students are expected to learn mathematics by attending to the proofs that their professors present to them, several important questions emerge. What, specifically, do we hope that students gain from reading these proofs? How successful are students at reaping these potential benefits? How should students engage with these proofs to increase their chances of learning from these proofs? What pedagogical actions can a teacher take to help students comprehend proofs more fully? The educational research literature on these issues is sparse, with especially little research on the second and third questions.

In this paper, I report on interviews in which nine mathematicians described their pedagogical practice of presenting proofs to their students. The data from this paper will shed light on why students are expected to read proofs, how they are expected to read them, and what types of things teachers can do to help, from the perspective of the professional mathematicians who teach these students.

2. Related literature

2. 1. Research on mathematicians' pedagogical practice

The relationship between researchers in undergraduate mathematics education is complex. Mathematicians are not only consumers of the research that mathematics educators produce, but they also frequently serve as critics of this research and are collaborators on research projects. In addition to these roles, mathematicians are sometimes the *subjects* of collegiate mathematics research in two respects. First, instructional goals of mathematics courses often include having students think and behave more like mathematicians (e.g., Schoenfeld, 1992) or participate in the same types of activities that mathematicians do (e.g., Rasmussen, Zandieh, Teppo, & King, 2005). However, as Stylianou (2002) notes, if bridging the gap between students' and mathematicians' practice is indeed a goal of mathematics instruction, it is necessary to have a clear and accurate understanding of how mathematicians actually think and behave when doing mathematics. Examining the behavior of mathematicians doing mathematics is useful toward developing this understanding.

A second way that mathematicians are studied as subjects in collegiate mathematics education research is to investigate their pedagogical knowledge and practice. There are at least three important reasons for doing this. First, Alcock (2009) notes that experienced mathematicians have an enormous amount of collective experience in teaching undergraduates mathematics; therefore investigating their pedagogical experiences provides a rich source of insight into students' mathematical reasoning and behavior. Second, Alcock (2009) argues that if the mathematics education community wishes to engage the mathematicians in discussions about their teaching, they need to understand the perspectives of mathematicians and take these perspectives seriously. Third, researchers in collegiate mathematics education often make suggestions for how mathematics pedagogy can be improved. For this research to make a practical impact, mathematics professors must choose to use these suggestions and must implement these suggestions effectively. However, mathematicians are unlikely to implement teaching suggestions if these suggestions are at variance with their pedagogical goals and beliefs or if the suggested pedagogy is perceived to be outside the norms of appropriate pedagogical practice. Consequently, understanding mathematicians' pedagogical goals and beliefs is important if the instructional recommendations of collegiate mathematics educators are to be widely disseminated. Despite the importance of understanding mathematicians' pedagogical practice, research in this area is rare. Although some recent volumes have investigated mathematicians' pedagogical practice in general (e.g., Burton, 2004; Nardi, 2007), few studies have examined their pedagogical practice with respect to proof.

In Weber (2004), I reported a case study of one professor teaching an introductory course in real analysis. Although the professor's instruction exhibited many of the characteristics of traditional instruction—including an emphasis on formalism and the use of a "definition-theorem-proof" format—two interesting findings emerged. First, the professor employed three distinct teaching styles as the semester progressed, beginning with an emphasis on logic, then moving toward a focus on proving techniques and procedures, and finally toward describing the semantic meaning of the concepts that were being studied. This finding suggests that traditional instruction may be more nuanced than is commonly assumed. Second, the professor’s pedagogical actions were based on a coherent set of mathematical and pedagogical beliefs. For instance, his initial emphasis on structure and logic was based on his belief that a proficiency in basic logical skills is necessary to make any progress in learning advanced mathematics. He subsequently emphasized proving procedures because his past experience indicated that students who were not shown these procedures would become frustrated and give up on the course.

Fukawa-Connelly (2005, 2007) studied two instructors teaching abstract algebra and reported similar findings. A self-reported “traditional” professor delivered instruction more nuanced than the definition-theorem-proof format, and frequently included examples and informal arguments in her lectures (Fukawa-Connelly, 2007). When students struggled with complicated problems, the professors would sometimes show students how the problem could be reduced to a standard procedure, thereby allowing students to avoid the conceptual issues that the problem was designed to elicit (Fukawa-Connelly, 2005).

Alcock (2009) conducted a series of interviews with five mathematicians about how they taught a transition-to-proof course. The mathematicians collectively discussed four types of proof-related skills that they were trying to teach: instantiating mathematical definitions and claims (i.e., constructing example objects that satisfied initial conditions), structural thinking (i.e., syntactic skills such as using the structure of a definition to outline a proof), creative thinking, and critical thinking. However, the mathematicians’ pedagogical techniques centered almost entirely on developing students’ structural thinking. Indeed, several of the participants in Alcock’s study expressed concern that they were not sufficiently developing students’ skills in the other respects and were painting a misleading picture that proving was a purely logical and procedural process. At the end of her article, Alcock remarked that two of the five mathematicians expressed doubt that all students were capable of learning how to prove.

Harel and Sowder (2009) conducted interviews with 22 mathematics faculty members about their pedagogical content knowledge with respect to proof. One interesting finding was that not all faculty members believed it was important for mathematics majors to master. “Perhaps surprisingly, university faculty do not uniformly seem to see the deductive proof scheme as an objective for all students, but only for the ablest” (p. 282). The professors collectively indicated that proof was a significant obstacle for most students, in part because they did not see any value in proof and especially because students did not understand or appreciate the role of definitions in mathematics. Similar to Alcock’s (2009) interviews, Harel and Sowder found that mathematicians did not seem to have a well-developed pedagogical arsenal for teaching students about proof. Harel and Sowder remarked that the professors seemed more concerned about having students understand particular proofs, rather than understanding and appreciating the role of proof in general. Finally, also consistent with Alcock’s (2009) study, Harel and Sowder (2009) found that some professors doubted that all students were capable of learning proof.

2. 2. Purposes of proof

To clarify what we expect students to gain from reading proofs, it is helpful to consider why mathematicians prove at all. Building on the philosophical work of Steiner (1978), Hanna (1990) distinguished between proofs that merely convince and proofs that also explain. Proofs that convince are designed to demonstrate to the mathematical community that a result is definitely true. Proofs that explain help the mathematical community understand *why* a result is true. Hanna (1990) and others have argued that mathematicians value proofs that explain more than proofs that only convince (e.g., Alibert & Thomas, 1991) and that explanatory proofs are more appropriate for mathematical classrooms (e.g., Hersh, 1993).

In an influential article, de Villiers (1990) listed five functions that proof serves within the mathematical community. In addition to conviction and explanation, proofs can be useful for facilitating communication by providing mathematicians with a common language to share and critique ideas, discovering new theorems, and systematizing a mathematical theory within an axiomatic framework. Consistent with de Villers' (1990) claim that proofs can be used as a tool for discovery, some researchers have advocated that proof can be useful in introducing students to new mathematical methods (Hanna & Barbreau, 2008; Weber, 2010a) that could potentially be applied in other contexts to discover new theorems. With regards to using proof to systematize mathematical theories, there is reason to doubt that this could be accomplished in most mathematics classrooms. As Harel and Sowder (2009) argue, students will only appreciate the need for clearly explicated definitions and axioms if they hold axiomatic proof schemes[1], and most mathematics majors do not think about proof in this manner (e.g., Harel & Sowder, 1998). Because of this, students are likely to think proofs that systematize theories have little value (Weber, 2002).

[1] An individual holds an axiomatic proof scheme "when a person understands that at least in principle that a mathematical justification must have started with undefined terms or axioms" (Harel & Sowder, 1998, p. 273).

In general, empirical studies suggest that students and high school mathematics teachers (e.g., Healy & Hoyles, 2000; Knuth, 2002) fail to recognize these various roles of proof, with many believing that the only function of proof is to provide conviction. One goal of this paper is to discuss mathematicians' views of what their students can gain from reading proofs in their classrooms.

2. 3. The reading of proof

Although there has been extensive research on students' difficulties with the construction of proof, several researchers have remarked that comparatively little attention has been paid on the reading of mathematical proof (Hazzan & Zazkis, 2003; Selden & Selden, 2003; Mamona-Downs & Downs, 2005; Weber, 2008; Mejia-Ramos & Inglis, 2009). Most research on the reading of proof has focused on the extent to which students or teachers can determine if an argument constitutes a valid proof. Numerous studies demonstrate that both populations have difficulty with this task (e.g., Martin & Harel, 1989; Segal, 2000; Knuth, 2002; Selden & Selden, 2003; Alcock & Weber, 2005; Weber, 2010b) and several researchers have used such tasks to infer students' beliefs about proof and conviction (e.g., Martin & Harel, 1998; Healy & Hoyles, 2000; Segal, 2000). There has been little research on *how* students should read a proof for evaluative purposes (see Weber & Alcock, 2005; Yang & Lin, 2008) and, to my knowledge, no research on how students should read a proof in order to gain understanding or insight. One aim of this paper is to discuss how mathematicians would like their students to read proofs for these purposes.

2. 4. The presentation of proof

There is little research on how students and mathematicians choose to present proofs. In a bibliographic study, Meija-Ramos and Inglis (2009) analyzed all articles in the ERIC data base that listed "argumentation" or "proof", as well as "mathematics", as key words that appeared in seven widely read journals in mathematics education. Of the 131 articles they considered, not a single article concerned the presentation of proof.

Two non-empirical articles (that were not included in Meija-Ramos and Inglis' survey) suggest alternative formats for presenting proofs of theorems. Leron (1983) proposes that comprehension of a proof might be facilitated if it is presented in a structured format, where the high-level ideas of the proof are given more prominence. Rowland (2001) endorses using *generic proofs*[2]—i.e., arguments that establish a general assertion holds true by carefully illustrating how it holds true for a specific example. This paper seeks to contribute to this literature by discussing the pedagogical techniques that mathematics professors use when they present proofs to their students.

3. Methods

3. 1. Participants

Nine mathematicians agreed to participate in this study. Initially, 19 professors at a highly ranked mathematics department at a large state university in the northeast United States were sent e-mails asking if they would be willing to participate in interviews about their pedagogical practice. These 19 professors were chosen either because I had previously interacted with them on matters dealing with mathematics education or because they had shown a friendly disposition to mathematics education in the past. Nine

[2] Generic proofs usually would not qualify as formal mathematical proofs.

of these 19 participants agreed to participate in this study. The process used to recruit participants likely led to professors who were especially reflective about their teaching; the interview studies by Alcock (2009) and Harel and Sowder (2009) also had this feature. The participants in this study were all tenured professors who were highly successful in their fields of study, including algebra, analysis, and differential equations. All regularly taught proof-oriented courses in undergraduate mathematics.

3. 2. Procedure

Each participant met individually with me for an audiotaped, semi-structured interview. The goal of these interviews was twofold: to investigate mathematicians' professional practice in reading the proofs of their colleagues and their pedagogical practice of presenting proofs in their classrooms. This paper focuses exclusively on the latter topic. A report on how mathematicians read proofs in regards to their professional work can be found in Weber and Mejia-Ramos (2011). To examine how and why they presented proofs in their classrooms, participants were asked questions that probed their views of what students were expected to learn when they read a proof, how they were supposed to engage in a proof to ensure this learning occurred, what difficulties students typically encountered with proof, and what pedagogical actions the participants used to alleviate these difficulties.

During the interview, participants were first shown a standard proof that $\sqrt{2}$ was irrational (given in the Appendix) and asked the following questions about the proof:

- In what classrooms, if any, would such a proof be appropriate to present?

- If you were to present this proof to the class, is there anything special that you would do? For instance, would you modify the proof, stress certain points, spend time explaining why one step makes sense, and so on?
- Why would you present this proof to a student?
- What do you think a student might gain from reading this proof?
- How long would you expect a student to spend studying this proof?

They were then shown a proof that real-valued bounded monotonic sequences necessarily converged (adapted from Abbott (2001) and given in the Appendix) and were asked the same five questions about the proof. The first proof was chosen because, from my point of view, its presentation emphasizes logic, and because the proof is usually included in transition-to-proof courses, at least in part, to illustrate proof technique. The second proof differed from the first in that it was more content-specific and I believe that it is presented in introductory real analysis courses, at least in part, so that students can appreciate why an important theorem in real analysis is true.

After this part of the interview, participants were asked the following questions:

- In your own mathematical work, do you sometimes read the published proofs of others? (If yes) What do you hope to gain out of reading these proofs?
- Is there a proof that you would consider a "must see" in one of your classes? Why?
- What do you think it means to understand a proof?
- What are some of the things you do when reading a proof to increase your understanding of it?

- When you teach upper level math classes, do you sometimes present proofs to your students or ask students to read proofs from textbooks? (If yes) What do you hope for students to get out of reading these proofs?
- What do you expect students to do when they read a proof?
- Do you expect students to read proofs differently than you would read a proof? Do you expect them to gain a different type of understanding from reading a proof?
- How can you tell if students understand the proofs? Do you ever test to see if they do understand the proofs?
- What are some difficulties that students have in understanding the proofs that are presented to them?
- Are there things that you do in your classes so that students might understand proofs better? (If yes) Why do you think these things might help students understand the proof better?

If participants said something that I found to be unclear or interesting, I would ask them to expand on these issues. Each interview lasted between one and two hours.

3. 3. Analysis

The interviews were analyzed using the constant-comparative method (Strauss & Corbin, 1990). Each response from a participant was initially parsed into episodes so that each episode contained a discussion of a single idea. In general, an idea constituted a specific strategy that a participant used (a proof reading strategy or a pedagogical strategy for presenting proofs to students), a rationale for using a strategy, or a benefit that they perceive for engaging in a mathematical practice (for themselves or their students).

Similar episodes were grouped together and given preliminary categories and definitions. New episodes were placed into existing categories when appropriate, but also used to modify the names or definitions of existing categories or to create new categories.

4. Results

4. 1. General attributes of the participants

The participants in this study generally appeared thoughtful and reflective about their pedagogical practice. Each, for instance, expanded at length on students' difficulties with proof and most described pedagogical strategies that they used when presenting proofs to their students. Nonetheless, many of the participants had not given explicit thought to many of the questions that I asked them. All of the participants gave extended pauses in their interviews, in some cases lasting longer than 30 seconds. At one point in the interview, M8 remarked, "I'm sorry. I'm not frustrated at your questions. They're good questions. I'm frustrated that I have been teaching for almost thirty years and I don't have answers to your questions". Similarly, M6 commented, "you know I really find this difficult because I haven't thought about this particular question so I'm not really satisfied with the answers I'm giving but that's all I've got".

There were several instances in the interviews when shortly after presenting a teaching strategy that the participant used, he or she recognized that what they had been doing was not effective or appropriate. For instance, when asked how she wanted students to understand a proof, M4 said, "First I'd like them to understand it line by line. [pause] That's probably the wrong advice to give. What I should want them to do is try it in a special case". Similarly, when asked how he assessed students' understanding of a proof, M6 responded, "That's a good question. You know, sometimes, I ask a student to do a proof on a test, and they do it, it doesn't necessarily indicate understanding. So that's a good point, they might have just memorized how to do that particular proof. And not in any reliable way". In conducting interviews of this type, it is always possible that the interview process may change the participants' knowledge and beliefs; there was evidence that this occurred in this study.

Finally, on some issues, there was not a high level of agreement between the participants. Indeed, in some cases, participants' responses to questions were contradictory. For instance, M6 explained that the proof that √2 was irrational was important since it was the first proof students typically saw "with real mathematical content". In contrast, M5 thought the only value of this proof was "showing how to write a proof" because "there aren't really any mathematical ideas here". Hence, the excerpts from an individual participant might not be representative of the responses of the other participants. To address this, I describe how many of the nine participants provided a response of each type.

4. 2. What can students gain by reading a proof?

When asked why they presented proofs to students or what they expected students to gain from this, the participants provided a wide range of responses, which are summarized in Table 1. Four categories of responses were mentioned by three or more participant: (a) illustrating a new proof technique, (b) some proofs are culturally important and students should be exposed to them, (c) exposure to proving in general is important, and (d) explaining why a theorem is true.

***** Insert Table 1 About Here*****

4. 2. 1. Illustrating a proof technique

Six participants indicated that they presented students with proofs to illustrate how to prove a statement with the hope that students could use similar reasoning in their proof-writing in the future. When asked why he would show a proof that √2 was irrational, M6 responded:

> M6: [long pause] That's really a tough question. […] This is like the first time I'm thinking about that particular question. I would say that there are these sort of, you know, you build up your family of examples of proofs, and you sort of have this foundation of a bunch of examples of proofs that you then get to draw on later on. You know, I'm not exactly sure if I can encapsulate exactly what's in this proof. Certainly proof by contradiction is such a thing, but then, you know, arguments based on parity of numbers, which are also really common. And being able to, you know, reason in that way about things. So that's another thing, another tool that illustrates.

Similarly, when asked why he would show a proof that monotonic bounded sequences necessarily converge, M2 remarked one reason this proof was useful is because it provided an illustration of how to prove that a sequence converged:

> M2: Well, there's a couple of different reasons […] A lot of the course material is about convergence. First you do convergence of sequences. This is from chapter 1. Then in chapter 2 you do convergence of functions, and the convergence of functions discussion is very parallel to the discussion for sequences… and in fact, this particular results gets used in the book to prove the corresponding statement for functions. You don't need to prove it that way, but that's the way that the book sets it up. So, I would do it more because it comes up a lot of times.

4. 2. 2. Cultural reasons for presenting proofs

Five participants indicated that there were some proofs that students should be exposed to because they were historically or culturally important. When asked why he would present a proof that √2 was irrational, M1 initially responded:

> M1: Because this is certainly, from the point of view of mathematical culture, since this goes back more than 2000 years as *the* dilemma of Greek mathematics, the irrationality of the diagonal of the square, you know, you have to present it. (emphasis was the participant's)

Two participants indicated that they sometimes presented proofs that they believed their students were not ready to comprehend. For instance, in the excerpt below, M8 questions whether students should be learning about proof by contradiction in a transition-to-proof course, but nonetheless indicates that he would include a proof that the $\sqrt{2}$ was irrational.

> M8: Proof by contradiction is an important technique. Students need to learn it, but I don't think they should learn it when they are first exposed to proof. When you are doing a proof by contradiction, I find students are not really basing it on their intuition but are making a lot of deductions until they find something that's false. You know, it's easy to make a mistake doing these things. I'd rather have students doing direct proofs first […]
>
> I: So are you suggesting that maybe this proof [that $\sqrt{2}$ is irrational] might be inappropriate for [the transition-to-proof] course?
>
> M8: Well, the difficulty here is that it needs to be seen. I mean, I think it's part of culture in several respects.

Similarly, M3 notes that he sometimes presents proofs of important theorems, even though students might not be ready to understand them.

> M3: There's a purely cultural factor there, you know, "look, if you put all these together you can do this". It's sort of a "don't try this at home" kind of proof. And you know, the Fundamental Theorem of Calculus is another thing like that. I mean that comes at a totally premature time but it needs to be explained anyway because it's fundamental to the history of the whole subject.

Herbst (2002) illustrates how high school teachers, when teaching proofs in geometry, sometimes engage in practices that they know will not enhance students' understanding out of respect for the customs associated with the two-column geometry proof format. In this section, we observe something similar. Some participants believe that proofs of historically important mathematical results should be presented to students, even if some students are not in a position to fully comprehend these proofs.

4. 2. 3. Exposure to proving

Four participants indicated that one reason that they showed students proof was to provide them with exposure to the proving process. In these cases, there was no specific learning goal in mind, but rather get an incremental understanding of what the proof they are reading was about or more generally, in M6's words, to help "them have an appreciation for what the whole enterprise [of proving] is about". In expressing these views, M4 and M5 questioned whether there was a single moment when the meaning of a proof became clear.

> M4: I don't learn by great revelations from studying one proof. I kind of learn by piling up bits of understanding, and I think from my experience with students they tend to learn that way also.

> M5: There are some proofs that students aren't going to understand. I don't think understanding happens all at once […] Do we teach them how to prove in [the transition-to-proof course]? Probably not, but it's a start […] At least they are getting some exposure to proof.

> I: So you are saying that for some mathematical ideas, you are not expecting understanding to come all at once?

> M5: I really don't. That's not the way that students have learned, in my experience. If you show them a proof, they might just get some idea, some aspect of it, but not understand the whole thing.

The idea that students can learn about proof merely by observing proofs in their classrooms seems unrealistic, so it is important to note that the participants in this study did not appear to be making this claim. As Table 1 demonstrates, no participant presented proofs to students *solely* to expose them to the process of proving. Rather, my interpretation of the participants' responses is that they are claiming that students might only understand some aspects of the proofs they are reading; therefore having students view a large number of proofs provide them with more opportunities to appreciate the many nuances of proof. If this interpretation is accurate, then this perspective is not inconsistent with the way that some mathematics educators view mathematical learning. Dubinsky, Leron, Dautermann, and Zazkis (1994) suggest teaching group theory by using a holistic spray approach in which students do not master specific concepts one at a time, but are thrust into group theoretic situations in which they enhance their understanding of multiple concepts bit by bit. "They [students] keep coming at it, always trying to make more sense, always learning a little bit more" (p. 300). Repeatedly exposing students to proof seems consistent with this approach.

4. 2. 4. Understanding why theorems are true

Four participants claimed they presented proofs to students was for students to deepen their understanding of why an assertion was true. For instance, M2 claimed an important reason for presenting proofs was to provide students with "an understanding of what makes the situation tick". When asked to elaborate, he responded:

> M2: Why it is that you have certain relationships between the things that you're looking at? So, for example, you know, if you're looking at some situation, and there's like a hypothesis, just fooling around. You want them [students] to understand why you need that hypothesis, so you want them to, on the one hand, to look at a situation where the hypothesis doesn't hold, and what you're looking at isn't true. On the other hand, you want to show why that hypothesis really is helping you, and to see why that hypothesis really is helping you, it helps to read the proofs.

Perhaps surprisingly, only one participant mentioned demonstrating that a theorem was true as a reason for presenting proof to his students (although several participants listed this as an important reason for reading the published proofs of others in their own mathematical work). In fact, in discussing the proof that √2 was irrational, M4 lamented that her students only viewed proof as providing conviction and this inhibited them from gaining understanding from the proofs that she presented to them.

> M4: My students never get anything out of it [the proof] at all, because they *know* the square root of 2 is irrational, they've been told that by authorities many times, and they have not yet got the idea that a proof is there to deepen understanding. They still think that a proof is there to establish truth, or perhaps to earn points on tests. (italics were M4's emphasis)

4. 3. Presenting proofs

As noted earlier in the paper, there has been virtually no research on how mathematics professors traditionally present proofs to their students. In this study, participants were asked what "special" things they would do if they were presenting a particular proof that √2 was irrational or that monotonic bounded sequences converged. There are two caveats that should be kept in mind when interpreting this data. First, because the data comes primarily from participants' discussion about two specific proofs, the data may only represent a subset of mathematicians' pedagogical strategies when presenting a proof. Asking about other proofs may have elicited different types of responses. Second, the participants were not observed actually presenting these proofs to a class of students and it is possible that there is a disconnect between what the participants claimed they would do and how they actually taught. The participants' comments about proof presentation are summarized in Table 2.

***** Enter Table 2 About Here *****

4. 3. 1. Presenting examples

Five participants claimed they would accompany a proof that they presented with an example[3]. Four of these participants claimed they would compare the example that they were considering with the proof they were presenting. They would, in essence, use the example to transform the formal proof they were presenting into a generic proof (in the sense of Rowland, 2001)[4]. For instance consider the two excerpts below:

> M4: Well, you know, I often, with such proofs I like to accompany the proof with an example. I’m always referring back to the example so that they can sort of have a model of a concrete instantiation of the steps of the proof.
>
> I: Oh, so you’d have a particular monotonic bounded sequence…
>
> M4: Yeah, you know, I’d have 3/4, 7/8, dot dot dot, and then try to, each time you do a step, you sort of do the step on the example at the same time. So that’s one thing that I might do.
>
> M5: Another thing I will do when showing a hard proof is to have a particular example side by side with the proof.

4. 3. 2. Drawing a diagram

Five participants mentioned that they would supplement the proof that monotonic bounded sequences necessarily converge with a diagram. To illustrate, when M2 read the proof presented in the Appendix, he remarked that he had recently proven a similar theorem in an introductory real analysis class. In describing how he would present the proof, he claimed that he would first unpack the hypotheses of the proof, which is reminiscent of Selden and Selden’s (2009) *formal-rhetorical* reasoning in writing a proof

[3] These comments were made in relation to the proof that bounded monotonic sequences converge or when speaking about proof presentation in general. The proof that √2 is irrational does not lend itself to example presentation.

[4] There is an important difference between these mathematicians’ comments and Rowland’s recommendations. These mathematicians would use generic proofs to *accompany* a formal demonstration, whereas Rowland would *precede* the formal presentation with a generic proof, or use the generic proof *in lieu* of a formal demonstration.

in which much of an undergraduate proof can be written by using the logical structure of the theorem being proved. Then he would draw a diagram

> M2: You unpack the hypothesis. And by the time you unpack the hypothesis, you're halfway through the proof that you see right on this page. And then what I would is I would draw a little picture to sort of give them the idea what's really going on, to suggest what it is that they're trying to show. And then I see here on this page, it seems reasonable to claim that blah… that's what the picture would be, is to make it reasonable that that should be the thing they should put…
>
> I: Right, so you're not pulling the "s" [the variable in the proof that is defined to be the supremum of the set of terms in the sequence] out of thin air?
>
> M2: There's an "s" in here, that's right. The way that this proof is written on the page, the "s" does kind of look like it's coming out of thin air. But if you draw a picture. You draw a horizontal line, that's the bounded part. And then what you do is you sort of keep lowering that thing as far down as it will go. And that's the "s" that's in the proof. So there's this visual "just drop the ceiling as far as you can". And then you say ok, so you want to prove that the limit is that guy.

Other participants mentioned that they would draw a picture prior to the proof, either to illustrate what the theorem was asserting, why it was true, or why the hypotheses in the theorem were necessary. A representative response is provided below.

> M1: Oh I certainly would draw a picture, of course, I mean I think most instructors would, because even though they know a picture is not going to prove it, it helps the intuition to understand this whole notion of why the monotonicity assumption is important.

4. 3. 3. Review relevant facts or concepts prior to presenting the proof

Two participants, M3 and M4, noted the question of how they would present a particular proof is difficult to answer, because how they would present each proof is dependent upon how the class was taught up until that point. M4 claims that she would only present a proof to her students if they had previously had time to absorb all of the concepts up until that point.

> M4: What's a little difficult about answering the questions is that I would only present this at a point at which I have already had time to discuss all these concepts, so that I feel like they've internalized… we've done lots of examples of supremum, and we've done lots of examples, so that I can refer back to those to try to get them to think in this way.

Here M4 is arguing that students should have sufficient experience with the concepts involved in a proof, including the consideration of many examples of those concepts, prior to reading proofs about those concepts. She would refer back to her previous examples in presenting the proof to help them think about the proof in a productive manner. Other participants suggested that they would review particular facts and concepts prior to presenting the proof. In the excerpts below, M1 discusses doing this with the proof that $\sqrt{2}$ was irrational and M5 does the same with the proof that monotonic bounded sequences converge.

> M1: The two things that I would stress is first that you have to review some basic facts about arithmetic. Which are in fact highly non-trivial, namely that you have factorization, prime factorization of integers, right?

> M5: This proof involves monotone and bounded sequences, so I would go over what monotone meant and what bounded meant. Here they introduce the idea of sup. I'd go over what sup meant. Maybe give some example sets. I'd draw the sequence on the real number line so students got a sense of what the sequence looked like and what was going on.

4. 3. 4. Emphasize new features in a proof

Four participants noted that if they presented a proof that introduced a technique or an idea for the first time, they would spend a great deal of time emphasizing that aspect of the proof. M3 noted the proof that monotonic sequences converged was likely the first time that his students had seen the completeness axiom applied in a non-trivial way, or more generally, proofs about significant mathematical content that depended heavily on a non-trivial axiom. Consequently, he would emphasize why the completeness axiom was necessary in the proof and how it was applied. M2 and M8 stressed that their presentation of √2 would focus on the use of proof by contradiction, which was the central idea in that proof that students likely had not seen before. Indeed, M2 argued that a central goal that he had when presenting a proof was to get students to focus on the "right part" of the proof; in his experience, students often concentrated on the algebraic manipulations in the proof that √2 was irrational rather than on how the beginning and the end of the proof was structured.

4. 4. Students' reading of proofs

The participants all claimed that reading a proof was a lengthy and complicated process and that, in general, students did not appreciate the complexity of the process. When asked how long it would take to read the proofs in this study, responses varied. One participant, M1, felt that students could understand the proof that √2 was irrational in five minutes. The other participants' responses ranged from fifteen minutes to two hours. Two participants, M5 and M7, remarked that some students would not be capable of understanding the proof that monotonic bounded sequences converged no matter how long they spent studying the proof.

Participants expressed concern that students did not appreciate the complexity of the process of proof reading, as is illustrated in the excerpts below:

> M1: One of the things that students are *not at all* accustomed to is the notion that once you start reading mathematics, serious mathematics, you can't read it without a pencil in hand and that serious mathematics is encoded in a kind of short hand, and every sentence has to be decoded. (emphasis is the participant's)

> M9: I find students read a proof like they would read a newspaper and it's impossible to understand proofs that way.

Participants' comments on what *specifically* they would like students to do when reading a proof and what pedagogical actions they took to assure that this would happen were sparse. M2, M3, and M7 commented that it would be desirable if students drew pictures to understand the proofs they were reading. As M2 notes, "If the student is smart enough to draw a picture, then I'd say the student would get the main idea of the proof. If the student is not smart enough to draw a picture, then it looks like a proof that comes out of nowhere". However, none of these participants could describe pedagogical actions that they took to encourage students to do this, aside from modeling the construction of a diagram in their proof presentations in class.

M1, M4, and M6 mentioned that it would be ideal if students related proofs to an example while they were reading the proof, or considered an example to help them understand an aspect of the proof that they considered to be problematic. M6 claims that he not only models this process for his students, but explicitly tells students that they should be doing this when they read proofs on their own, both in his lectures and in the course notes that he distributes for students. M1 claims that he will do this with students if he is working with them individually, but that it is difficult to communicate how examples should be considered in a course lecture. During the course of her interview, M4 realized that while she desired students to consider examples when reading proofs, her lectures encourage students to focus on reading the proof at a line-by-line level, which is not really what she wants the students to do.

> M4: First I'd like them to understand it line by line. [pause] That's probably the wrong advice to give. What I should want them to do is try it in a special case. To do the two-column gizmo of the general proof in parallel with a fairly generic special case.

Four of the participants, M3, M4, M7, and M9 claimed they emphasized line-by-line reading in their instruction, even though all of them described the processes that they themselves used to understand a proof went beyond simply understanding the meaning of each assertion in a proof and the proof's logical structure.

4. 5. Assessing students' understanding of a proof

Table 3 summarizes participants' responses to how they would assess the extent to which students understood the proofs that they were given. The most common response, provided by five participants, is that students would be asked to adapt the method in the proof that they observed to prove a similar theorem. For instance, to assess whether students understood the proof that $\sqrt{2}$ was irrational, they would be asked to produce a proof that $\sqrt{3}$ was irrational. Several of the participants acknowledged that this did not assess whether students understood the proof deeply. M3 said "we would of course test them on their ability to simply make similar proofs, and it's not very robust. I mean, there's this sort of alphabet soup problem [that students might write a similar proof by blindly changing the names of variables within the proof]" but concluded, "that's the minimal level of understanding and that's the one we like to test".

M1 and M6 claimed that he would sometimes ask students to reproduce a proof on a test that he presented to them in class. Both realized this was only assessing a shallow understanding. M6 noted, "I ask a student to do a proof on a test, and they do it, it doesn't necessarily indicate understanding. So that's a good point, they might have just memorized how to do that particular proof. And not in any reliable way". M1 made similar comments.

> M1: In that case, I guess what I'm asking for as to whether they have understanding is not a terribly profound understanding. I'm basically asking if they can reproduce the proof that I gave.

M4 and M7 claimed that they did not evaluate students' understandings of the proofs that they presented in any reliable way.

***** Insert Table 3 About Here *****

4. 6. Should all mathematics majors learn about proof?

Although the aim of this study was not to discuss mathematicians' views about whether all students should learn proof or if all students were capable of doing so, these issues did emerge in several of the interviews. The university in which this study was situated offered two courses in introductory real analysis. The first is a 400-level course entitled "real analysis" that is reserved for the most able mathematics majors who are considering pursuing a graduate degree in mathematics. The second is a 300-level course entitled "advanced calculus" that is designed for the rest of the mathematics majors. I did not bring up these different courses during the interviews, but several of the participants did. While the participants who made this distinction all believed that rigorous proof was appropriate for the real analysis course, several questioned whether it was appropriate for the advanced calculus course. These participants doubted that many of the students in this class would have the ability to understand these proofs. Two representative comments of this type are provided below:

> M1: [300-level advanced calculus] is a course I never want to teach. I never taught it and I never want to teach it. […] I think that getting the ideas across in [advanced calculus] is very difficult for students who have very little aptitude. I'm also a musician and I don't want to try

> to teach people music who are tone-deaf and I think many of our students wind up being, from the point of view of proofs, the functional equivalent of being tone-deaf.[5]

> M7: When I teach [300-level advanced calculus], I've found that, at most, half the students are going to understand the whole enterprise of proof. Half, no matter what I try to do, won't.

These comments are consistent with comments made by some of the mathematicians in Alcock (2009) and Harel and Sowder's (2009) interviews. Interestingly, a mathematics professor in Harel and Sowder's (2009) study, like M1, also wondered if teaching proof to some students was like trying to teach music to someone who was tone deaf (p. 288).

Three participants questioned whether proof should play such a prominent role in courses like the 300-level real analysis course. As M5 noted, "I'm not sure what most students get out of [reading a proof]. There are a lot of ways to understand mathematics. You can get a sense for why things work, see how things work in particular examples, it's not just proof. There are more ways to understand mathematics than that. But proof is all that we teach". M2 desired that the activity in real analysis be more concrete, focusing on ideas such as why particular approximation methods converge, rather than an axiomatic treatment of abstract sequences, series, and functions. After he read the proof that monotonic bounded sequences converged, he acknowledged that the proof was central to real analysis, but wondered aloud, "This proof, I can't see why some students would care about it. I guess this might be useful for some approximations, but very few approximations are bounded. Not all of our students are going to become mathematicians". The finding that some mathematicians questioned whether exposure to

[5] M1 did note that with hard work, he had developed the proving capabilities of "tone deaf" students in the past, but not to the point that these students were fluent with reading or writing proofs.

rigorous proof was appropriate for all mathematics majors was also found in Harel and Sowder's (2009) study.

5. Discussion

In this section, I organize the data around three issues that are important components of essential teaching: (a) what do the mathematicians participating in this study expect students to learn when they present proofs to their students, (b) what pedagogical actions are useful toward achieving these goals, and (c) how might a student reason and behave to ensure that the desired learning occurs. These questions are central questions to many research programs whose aim is to design effective instruction, including APOS theory (Asiala et al, 1998) and much of designed-based research (as characterized by Cobb et al, 2003), although there is considerable variation among the methodologies that researchers use to address these questions.

There is a large consistency between mathematicians' and mathematics educators' goals for presenting proofs to students. In general, the participants favored presenting proofs for explanatory purposes with only one participant listing providing conviction as a primary reason for presenting proofs to students. Likewise, many mathematics educators have argued that in mathematics classrooms, it would be more useful to use proof as a tool to explain than to convince (e.g., Alibert & Thomas, 1991; Hanna, 1990; Hersh, 1993). The majority of participants also claimed to use proof as a way of illustrating reasoning or proving techniques, something that mathematics educators have recently argued was important (e.g., Hanna & Barbreau, 2008; Weber, 2010a). The participants also claimed they would present proofs to provide students with exposure to proof or for cultural reasons (primarily via the transmission of cultural artifacts). Some mathematics educators might object to these practices, although perhaps they might be useful for purposes of enculturation.

The participants appeared to lack an arsenal of pedagogical strategies to achieve their goals. They did suggest strategies to improve the comprehension of proofs that they presented, including drawing diagrams, analyzing specific examples, and recalling facts and concepts that were involved with the proof. This finding corroborates Fukawa-Connelly's (2007) claim that, for professors of mathematics, proof presentation is a complicated process that is not as formal as some caricatures of university mathematics teaching. However, most participants seemed to provide little guidance to students on how to engage in the complicated process of reading and comprehension of proofs and, by their own admission, lacked adequate methods for assessing students' understanding of a proof. Having adequate assessments of students' proof comprehension can be useful for professors to evaluate their teaching and, as Conradie and Frith (1990) argue, can provide students' direction into how a proof should be read. Currently the mathematics education literature does not provide adequate assessment instruments for measuring students' understanding of a mathematical proof at the university level, although my colleagues and I have recently proposed a model to accomplish this (Mejia-Ramos et al, in press).

The participants said little about how they expected students to read proof. Some participants contended that students should draw a diagram or consider a specific example when reading a proof but, aside from modeling these processes in their proof presentation, they generally did not take pedagogical actions to increase the likelihood that students would behave in this way. Although the participants lamented that students did not appreciate how time-consuming and complex that proof reading was, few cited pedagogical strategies that they used to help students overcome this difficulty. Perhaps one reason for this difficulty is that the participants themselves had only a *prescriptive* understanding of their proof-reading strategies but not a *descriptive* understanding—that is, they could name specific strategies that they engaged in but their knowledge of how they actually implemented these strategies was largely tacit. The strategies mentioned by the participants, considering examples and diagrams, are also common heuristics in problem-solving. The problem-solving literature demonstrates that simply making students aware of these strategies or modeling their implementation is not sufficient to improve students' ability to solve problems (Begle, 1979; Schoenfeld, 1985). Schoenfeld argued that this is because the implementation of these strategies is complicated and dependent on one's knowledge of the mathematical domain being studied. It seems that a similar result would hold in using diagrams and examples to understand proofs. Regardless of the cause, there is a paucity of research on the processes that students should engage in when reading proofs (see Inglis & Mejia-Ramos, 2009; Weber, 2008) and more research in this area is needed.

Finally, like the mathematicians in Harel and Sowder's (2009) study, several participants in this study questioned whether it was appropriate to convey information to *all* students via mathematical proof. Perhaps less able students who do not intend to become mathematicians do not need to know why important theorems in abstract mathematics are true (or even know these theorems at all) or perhaps they could get a sense for why these theorems were true in a medium other than formal proof. It is not clear that students who have serious difficulties comprehending proofs or students who are more interested in applying mathematics to other disciplines are best served by the current emphases in mathematics courses. The issues of what mathematics would be most appropriate for these students and of whom rigorous proof is appropriate for would be interesting topics of future discussion.

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Appendix.

Proofs used in this study.

Theorem. $\sqrt{2}$ is irrational.

Proof. Suppose $\sqrt{2}$ is rational.

Then there exists integers a and b such that $\sqrt{2} = \frac{a}{b}$.

Assume $\frac{a}{b}$ is reduced so a and b have no factors in common.

Now $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$.

Since a^2 is even, a is even.

We can therefore write $a = 2k$ for some integer k.

Then $2b^2 = a^2 = (2k)^2 = 4k^2$.

Hence, $b^2 = 2k^2$ so b^2 is even.

But this implies b is even, contradicting the assumption that a and b have no common factors.

Statement: If a sequence is monotone and bounded, then it converges

Proof.

Let (a_n) be monotone and bounded.

To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit.

Let's assume that the sequence is increasing (the decreasing case is handled similarly) and consider the set of points $\{a_n : n \in \mathbf{N}\}$.

By assumption, this set is bounded, so we can let $s = \sup\{a_n : n \in \mathbf{N}\}$.

It seems reasonable to claim that $\lim(a_n) = s$

To prove this, let $\varepsilon > 0$.

Because s is the least upper bound of $\{a_n : n \in \mathbf{N}\}$, $s - \varepsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \varepsilon < a_N$.

Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$.

Hence, $s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon$, which implies $|a_n - s| < \varepsilon$ as desired.

from Abbot's (2001) real analysis textbook, *Understanding Analysis*.

Table 1. Participants' reasons for presenting proofs in their classes

Reason	**Participants**
The proof illustrates a new proving technique	M1, M2, M4, M5, M7, M9
The proof is important for cultural reasons	M3, M4, M6, M7, M8
Exposure to proof is important	M1, M2, M5, M9
A proof illustrates why a theorem is true	M1, M4, M6, M8
The proof provides a deeper understanding of the content of the theorem	M1, M3
It's easier to remember a theorem if you've seen it's proof	M4
A proof illustrates the latent complexity of a mathematical statement	M3
A proof establishes that a theorem is true	M8
A proof introduces the students to a new way of thinking	M3

Table 2. Pedagogical techniques that participants claim to use when presenting a proof

Presentation technique	**Participants**
Accompany the proof with an example	M1, M4, M5, M7, M8
Draw and use a diagram	M1, M2, M5, M8, M9
Emphasize new features of a proof that were not present in previous proofs	M1, M2, M3, M8
Recall relevant facts prior to presenting the proof	M1, M5, M9
Spend time on steps that will be problematic for students	M5, M9
Use a dialog format so students are involved with the proof construction	M2

Table 3. Participants' responses for how they assess students' understanding of a proof

Type of assessment	**Participants**
Ask students to use a proof to construct a similar proof	M2, M3, M5, M8, M9
Ask students to reproduce a proof	M1, M6
Does not assess students' understanding of proofs	M4, M7