Mathematical humor: Jokes that reveal how we think about mathematics and why we enjoy it

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Mathematics and humor

Paulos (1980) argued that there were deep connections between mathematics and humor; proof in particular is similar to humor in that both follow logic, patterns, rules, and structures and both require cleverness, playfulness, ingenuity, and sometimes, reductio ad absurdum. Rentaln and Dundes (2005) provided a catalog of mathematical jokes, where the authors use the jokes to understand mathematicians' identities and "even a clue as to the nature of mathematical thinking" (p. 24). For instance, Rentaln and Dundes cite the joke "How many mathematicians does it take to screw in a lightbulb? 0.999" as evidence that mathematics is largely a solitary endeavor (a position that I do not share as collaboration is quite common in contemporary mathematical practice). Yet as Aberdein (2013) observed, "the role of humor in mathematical cognition has seldom been addressed" (p. 231). He analyzed the classic joke about spurious proof methods— for instance "proof by vigorous hand-waving- works well in a classroom or seminar setting" — as a lens toward understanding mathematicians' practice. Other than the three references cited in this paragraph, I am not aware of others who have written about this subject.

In this essay, I will use recent advances in humor theory as a means to understand jokes about mathematicians, jokes about mathematical work, and why we enjoy visual proofs. Specifically, I will build on Hurley, Dennett, and Adams' (2011) theory of humor that is tightly tied to their theory of mind. These authors argue that what we find funny is critically dependent on the subconscious fallible inferences that we all make and the models that we form about how other people think. I argue, consequently, that by studying jokes about mathematics and mathematicians, we can gain insight into how we think about mathematics.

A theory of humor

I want to die peacefully in my sleep like my father, not screaming in terror like his passengers.
(Bob Monkhouse, cited in Carr and Greeves, 2006).

Hurley, Dennett, and Adams' (2011) theory of humor is based on the premise that it is evolutionarily advantageous to identify when false beliefs have entered our cognitive system. Because deliberate conscious reflection requires effort and is, for many individuals, unpleasant, we might not engage in the process of rooting out false beliefs if we were not rewarded for doing so. Hurley, Dennett, and Adams postulated that the reward we reap for identifying inconsistencies in our cognitive systems is mirth— so long as no harm is caused, we enjoy any insight that allows us to identify and fix an inconsistency. They also argued that jokes are possible because human beings can capitalize on this evolutionary process to deliberately induce mirth. In the joke above about the dying father, when we read, "I want to die peacefully in my sleep like my father", we naturally think of a serene scene of a man dying in his bed. This is a probabilistic inference— it is not actually stated in the joke— but it is a sensible inference to make. We are therefore jolted when presented with the very different situation of a man operating a vehicle who killed his passengers because he fell asleep behind the wheel. Because we are not harmed, we experience pleasure at this insight into an alternative interpretation.

Hurley, Dennett, and Adams (2011) further argued that it is possible to identify three different types of humor that induce mirth in different ways, which I will refer to as first person humor, third person humor, and unresolved semantic ambiguity. With first person humor, we experience mirth when we discover differences in the inferences we made using our intuitive schemes and those that we made using deliberate logic. With many forms of third person humor, we take pleasure in noticing inconsistencies or errors in somebody else's thinking. Frequently, we laugh at the inferences made by someone who has a deficient cognitive system when we see this individual drawing inappropriate or irrational inferences than differ from the interpretations that ordinary individual with common sense would make. With unresolved ambiguity, we experience mirth when we realize that the same stimulus can be interpreted in two different ways and each way makes sense; puns often fall into this category. I will argue that all three of these humor types can be found in mathematical humor. Further, I will distinguish between jokes about mathematical situations and jokes about mathematical work. With the latter, these jokes can reveal how we think about mathematics. Specifically, I argue that visual proofs can be viewed as cases of mathematical unresolved ambiguity and explains one of the reasons we enjoy mathematics.

First-person humor

There are 10 types of people in this world. Those that use binary notation and those that do not.

This joke rests in an obvious way on our unconscious (and again quite sensible) assumption that we are working in base-10 and therefore 10 means ten. But there are also first person jokes involving more extended mathematical work. Some of these take the form of spurious proof where the logic of the proof intuitively seems sound, but the proof itself leads to an absurdity. Consider the following argument:

(1) Assume $a = b$
(2) Then $a^2 = ab$
(3) $a^2 - b^2 = ab - b^2$
(4) $(a - b)(a + b) = b(a - b)$
(5) $a + b = b$
(6) Using the fact that $a = b$, by substitution, we obtain $b + b = b$.
(7) $2b = b$
(8) $2 = 1$

Rentaln and Dundes (2005) referred to spurious arguments of this type as "jokes", but in what sense do they qualify in relation to Hurley, Dennett, and Adams' (2011) theory? I propose that as we gain extensive experience studying mathematics, some routine manipulations become part of our intuitive interpretation schemes for doing and comprehending mathematics. We read the argument above as "doing algebra" and "doing the same thing to both sides" which we quickly and intuitively accept as unproblematic. The joke is that we thought that we intuitively found dividing both sides of an equation by $(a - b)$ to be a valid move and only upon reflection, induced by the jarring conclusion that $2 = 1$, did we realize that our intuitive inference is inaccurate.

This joke illustrates a truth about mathematical practice. As Grcar (2013) documented, published mathematical proofs sometimes contain errors. Devlin (1993) argued that there is a specific type of error that does not usually occur in the technically complex or novel parts of the proof. Reviewers are aware that their intuition cannot be trusted in these cases so they inspect these aspects of the proof carefully. Rather, according to Devlin (1993), published errors typically occur when mathematicians apply a familiar method in a way that seems intuitively permissible. Some reviewers do not always check every line in a proof if they believe the main methods of the proof to be sound (Mejia-Ramos and Weber, 2014; Weber and Mejia-Ramos, 2011). This is for good reason. The small risk of a fatal error in a proof may not warrant the investment of time to check every step of the proof. Similarly, students cannot be blamed for "doing algebra" and temporarily suspending their understanding of the equations being manipulated. If a student were to consciously attend to the meaning of each statement in an algebraic derivation and explicitly verify every assumption underlying the every algebraic manipulation that was performed, she would scarcely be able to complete her homework in calculus. So the spurious proof makes sense as a joke that draws attention to an inconsistency and provides an opportunity for insight: when we "cancel out terms" while "doing algebra", it is important to verify that we are not inadvertently dividing by zero. Because no one was harmed by this spurious proof, we experience mirth. If harm was done— if we published an invalid argument, students lost points on an exam, or a field of mathematics collapsed because of a spurious proof— we would not regard the situation as humorous.

To clarify this point, it is worth comparing our "joke" with other deficient arguments. Consider the spurious argument below:

(1) $3^2 + 4^2 = 5^2$
(2) $(3 + 4)^2 = 5^2$
(3) $7^2 = 5^2$
(4) $49 = 25$

There is nothing funny about this argument. It's simply foolish. No one with mathematical training would think that $3^2 + 4^2$ was intuitively equal to $(3 + 4)^2$. False proofs are funny only if there is some interpretation scheme by which they make sense. Compare the foolish proof with one that you might think is more humorous.

(1) $e^{2\pi i} = 1$
(2) $(e^{2\pi i})^i = 1^i$
(3) $e^{2\pi i * i} = 1$
(4) $e^{-2\pi} = 1$

Assuming that the reader is familiar with the fact that $e^{2\pi i} = 1$, every step of the proof appears to be sensible. Like the proof that $2 = 1$, this proof is also a joke with a moral. The ordinary rules of exponentiation do not hold for complex exponentiation.

Third person humor

A logician's wife told him, "go to the store and get a quart of milk. If there are eggs, get a dozen". The logician came home with three gallons of milk.

In third person humor, the unproductive inferences that we laugh at are often based on stereotypical shortcomings of some other group. In mathematics, such jokes are typically about mathematicians or, as here, specific groups of mathematicians like logicians. There are many variants of this joke with the same theme: logicians have an abnormal scheme for interpreting natural language when conditionals, conjunctions, and disjunctions are involved. In the joke above, we laugh because we can model the hypothetical logician's thought process and see that he made an inappropriate inference about his wife's intentions. An ordinary individual would realize his wife was asking for a dozen eggs. This joke might therefore be funny to non-mathematicians because it plays on the more general stereotype that mathematicians are smart but they lack common sense and that mathematicians have trouble behaving normally in social interactions. This is a common part of the joy of third person humor of this type: we feel superior to the characters in the humor. Jokes of the form, "why did [the oppressed group] do [abnormal or inappropriate behavior]?" usually have the punch line that reveals deficient or deviant cognitive processes of the oppressed group in question.

However, I would argue that this is often not the case with jokes about mathematicians. In order to appreciate the third person joke above, one would have to be familiar enough with logic to form a mental model of the logicians' thought processes that led to their inappropriate behavior. If the punch line was, "the logician came home with three loaves of bread", the joke would not be funny as we could not produce a model that would account for the logician's behavior. To clarify this point, consider a more "difficult" joke:

Three logicians walk into a bar. The bartender asks, "would you all like a beer?"
The first logician says, "I don't know".
The second logician says, "I don't know".
The third logician says, "yes".

I would argue that most of the fun here lies in is solving the riddle for why the third logician answered so definitively [1]. We laugh here not because we feel superior to the logicians, but because we believe their reasoning makes sense from a certain mathematical point of view. And this is a common phenomenon: the audience for jokes about mathematicians is usually mathematicians themselves— not many other people would be able to understand these jokes.

We also see third person humor in deficient mathematical work. For instance, consider the following:

$\frac{d}{dx}\frac{\sin x}{x}$
$= \frac{d}{dx}\frac{\sin \not{x}}{\not{x}}$
$= \frac{d}{dx}\sin$

[1] If this is not clear, consider the information that we can glean about the first and second logicians' preferences based on their responses.

$= \cos$

Mathematicians who have experience teaching calculus might see this as humorous, as it is indicative of some students' mindset to cancel furiously and not to think about the objects represented in the expressions that they are manipulating. This can be viewed as a joke with a moral to discourage novices in training from engaging in meaningless manipulations by illustrating it in a harmless way with a hypothetical student. This is less painful than deducting points from their actual work or directly insulting them.

There are also meta-jokes in which the humor follows from one violating expectations of the joke format. (e.g., "Three men walk into a bar. The fourth man ducks"). We see that in jokes about mathematical work as well.

$\frac{64}{16} = \frac{\not{6}4}{1\not{6}} = \frac{4}{1}$

$\frac{d}{dx}x = \frac{\not{d}}{\not{d}x}x = \frac{x}{x} = 1$

In the derivations above, we again see someone engaging in nonsensical algebraic manipulations. However, the results of these simplifications are not the absurdities that we would expect, but true mathematical assertions.

Unresolved ambiguity humor

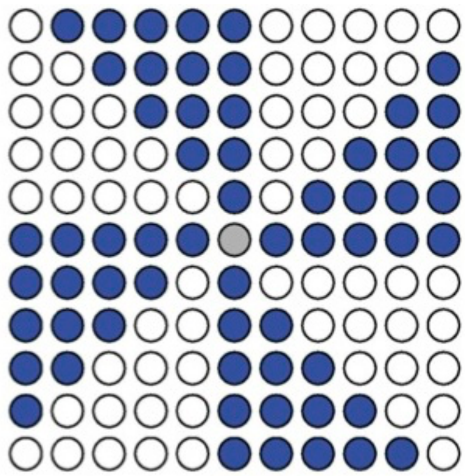
A cardboard belt is a waist of paper.

The third category of jokes occurs when the same stimulus can be interpreted in two different ways and each way makes sense. The pun above is one example of this. There are jokes (mostly bad ones) about mathematics involving unresolved ambiguity. For instance, consider the following from Rentaln and Dundes (2005):

Why didn't Newton invent group theory?
Because he wasn't Abel.

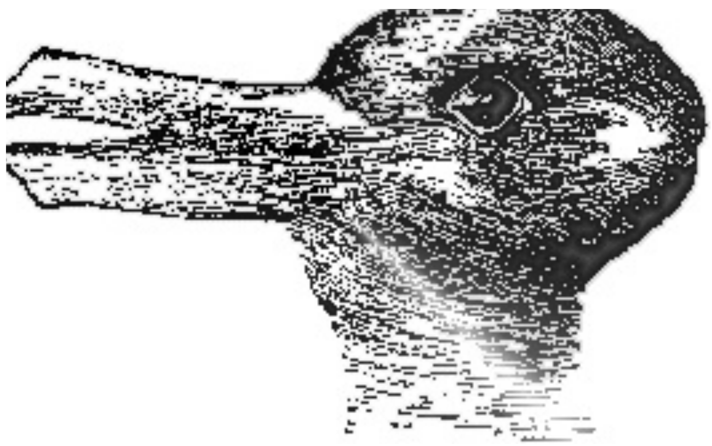
Why can't you grow wheat in $\frac{\mathbb{Z}}{6\mathbb{Z}}$?
Because it's not a field.

The common thread in jokes of this type is that the names of mathematicians or mathematical objects have other non-mathematical meanings in natural language. But what about instances of unresolved ambiguity that are applied to mathematical work? I contend the diagram below can be seen as similar to such a joke that for all odd integers n, $n^2 \equiv 1 (\mod 8)$ (from Nelsen, 2008).



I like this visual proof. And I suggest two reasons for this— and for the more general phenomenon that many mathematicians enjoy visual proofs. First, the diagram is like an unresolved ambiguity joke. When we see the diagram, as mathematicians, we desire a systematic way to count the number of dots in the diagram. The diagram simultaneously suggests two ways to do so. The diagram can be viewed as an n by n square and we have a trivial interpretation scheme for counting the dots in such a figure: there are n^2 of them. But the coloring of the dots suggests that the diagram can be interpreted in another way. There are eight congruent shapes plus a stand-alone dot in the center. Hence we can read the same diagram in two different ways—as representing n^2 dots or as representing eight congruent shapes plus one dot. If asked which interpretation is the correct one, we would respond that both interpretations are correct. In this sense, the diagram above has similarities to a mathematical pun. However, instead of stating a banality like a waste of paper is a waist of paper, we see an interesting mathematical truth.

Second, the diagram provides this insight suddenly. One has the experience— as with a joke— of seeing a situation in one way, then realizing that it can also be interpreted in another. This insight provokes pleasure. I thus like the proof not because it explained to me why the theorem was true or because it expanded my knowledge base in some way; I could easily give several short algebraic proofs for why n^2 is congruent to 1 (mod 8). I like the proof because I think it is interesting and clever. Given the popularity of Nelsen's (1993, 2000) *Proofs without words*, I do not think that I am atypical. What I suggest is that using two different interpretation schemes to see the same mathematical stimulus in two different ways is a natural source of enjoyment.



There is a reason why this is especially pleasurable in mathematics, more so than in puns or the famous drawing above that simultaneously can be viewed as a duck or a rabbit. The visual proof above can be viewed *generically* as a mere representative of a class of the arrays of squares whose sides have odd lengths (Leron and Zaslavsky, 2014). In the duck-rabbit drawing above, this is one particular picture. We would not expect to see a rabbit in future drawings of ducks that we encountered. Similarly, in the pun about Newton and Abel, we would not expect Newton to *never* be capable of some of the things that Abel accomplished. (The joke, "why didn't Newton study mathematics? Because he wasn't Abel" would make no sense). If we looked at the visual proof as merely showing that the specific number 121 was both 11^2 and congruent to 1 modulo 8, it would be a trivial and unsatisfying observation. It is that we can see the relationship displayed in the square as occurring in any square with an odd number of sides (and much of "getting" the visual proof is demonstrating to oneself that this is indeed the case).

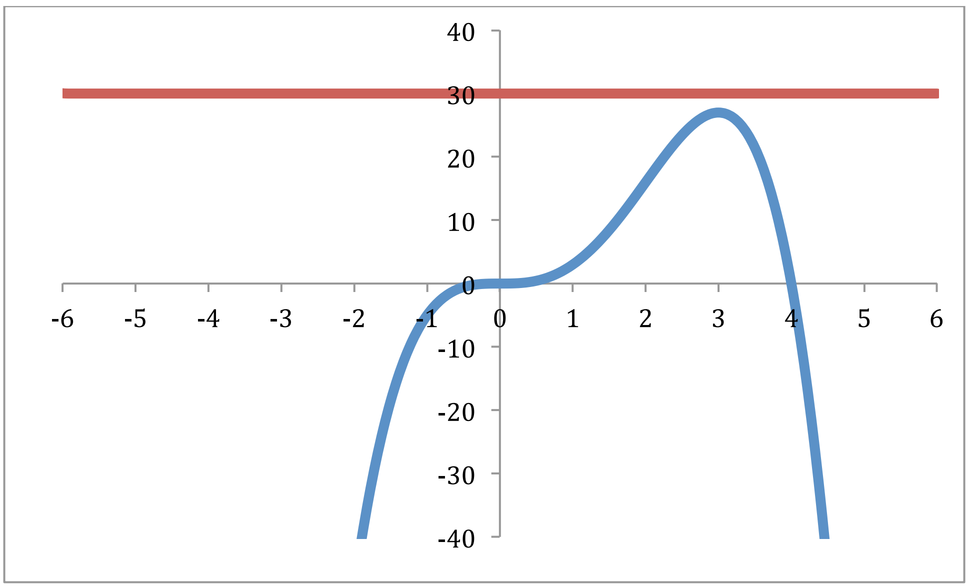
Are visual proofs better able to provide pleasure than symbolic ones?

The claim is that visual proofs are particularly enjoyable since we can make two different, yet equally valid, interpretations on the same picture.

Here it is worth noting that not all visual proofs have this property and, occasionally, some symbolic expressions do have this property. For instance, suppose I asked you to solve the following algebraic equation.

$$4x^3 - x^4 = 30$$

A visual proof that there are no solutions is given below.



The proof is illuminating as the natural intuition is to engage in algebraic manipulations to obtain a solution. However, this approach is difficult in this case as this involves solving a quartic equation. The approach suggested by the visual proof— use calculus and consider extrema (it is trivial to show that the global maximum of $f(x) = 4x^3 - x^4$ occurs at $f(3) = 27$)—can instantly be seen to be more productive. However, we are only applying one interpretation scheme when reading these graphs; namely that a solution for $f(x) = g(x)$ occurs in the instances where the two graphs intersect. Now consider the following claim.

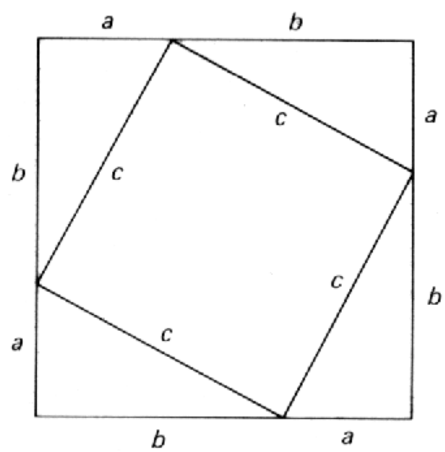
For any n objects, the number of subsets with odd cardinality is equal to the number of subsets with even cardinality.

A symbolic expression pointing to an algebraic proof is given below.

$$(1 + (-1))^n$$

Assuming n is a natural number greater than zero, applying an arithmetic interpretation to this expression trivially yields $0^n = 0$. Applying a binomial interpretation to this expression yields $\sum_{i=0}^{i \leq n/2} \binom{n}{2i} - \sum_{i=0}^{i<n/2} \binom{n}{2i+1}$. When we set the latter expression equal to 0, the desired result can be deduced. In this sense, the symbols $(1 + (-1))^n$ has a similar role as the diagram showing that for odd n, $n^2 \equiv 1 (\mod 8)$. It acts as a fulcrum to which two different interpretations can be applied.

It is not that all visual proofs involve unresolved semantic ambiguity. Rather, visual proofs are particularly suited as stimuli for multiple interpretation schemes for two reasons. First, perceiving mathematical structures involves seeing the relationships between different mathematical objects. While the existence of such structures can often be deduced when expressed as verbal-symbolic sentences, it is often much easier to perceive such relationships when they are expressed pictorially (Larkin and Simon, 1987). For instance, although we can deduce that the segments AB, AC, and BC form a triangle (assuming A, B, and C are not co-linear), it is more transparent if we are simply shown a picture of triangle ABC, which as Macbeth (2014) notes, simultaneously symbolizes both an empirical and theoretical triangle. In the picture proof that for odd n, $n^2 \equiv 1 (\mod 8)$, we can define how to construct each of the eight congruent shapes in the diagram and then deduce that they are indeed congruent, but it is much easier to simply see this in the picture. Second, semantic ambiguity usually involves seeing two different structures in the same picture, where the objects in the picture are simultaneously part of two different structures. With the exception of the middle dot, each dot in the picture proof can be interpreted as both part of one of the eight congruent pieces and part of the larger square. Pictures in two-dimensional space are better suited toward displaying multiple relationships than verbal-symbolic expressions that are expressed linearly. To illustrate, consider the following classic picture-based demonstration of the Pythagorean Theorem:



$$(a+b)^2 = c^2 + 4(\tfrac{1}{2}ab)$$
$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

Note that in this picture, we see a part-whole relationship being expressed. The diagram can be viewed simultaneously either as one large square or as a smaller square with four smaller triangles. Each of the segments in the diagram is serving double duty. The legs of the right triangles also form the perimeter of the larger square. The hypotenuses of the right triangle also form the perimeter of the smaller square.

Conclusion

In this paper, I explored the link between humor and mathematics and what this can tell us about we think about mathematics. Mathematical fallacies are linked to first-person humor. Both capitalize on us intuitively making an inference that is later shown to be problematic. Jokes about mathematicians and some types of erroneous argument are linked to third person humor. Both capitalize on our understanding of inferences made by others. Visual proofs are linked to puns with unresolved semantic ambiguities. Both involve providing a stimulus that can be interpreted using two different schemes, with each scheme yielding sensible, yet semantically different, interpretations. Because mathematical truth is objective and a-contextual (or at least we act *as if* it is), in the case of visual proof, both interpretations are valid and when compared to one another, can yield interesting mathematical truths. Hence, the pleasure that we experience with the unresolved ambiguity of visual proofs is heightened by the fact that through these representations we can also perceive the unity of mathematics.

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