Mathematics majors' beliefs about proof reading

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Abstract. We argue that mathematics majors learn little from the proofs they read in their advanced mathematics courses because these students and their teachers have different perceptions about students' responsibilities when reading a mathematical proof. We used observations from a qualitative study where 28 undergraduates were observed evaluating mathematical arguments to hypothesize that mathematics majors hold four specific unproductive beliefs about proof reading. We then conducted a survey about these beliefs with 175 mathematics majors and 83 mathematicians. We found that mathematics majors were more likely to believe that when reading a good proof, they are not expected to construct justifications and diagrams, they can understand most proofs they read within 15 minutes, and understanding a proof is tantamount to being able to justify each step in the proof.

Keywords: Beliefs, proof, proof reading, proof comprehension, undergraduate mathematics education

1. Introduction

In advanced mathematics classrooms at the university level, mathematical content is primarily conveyed to students by having the professor present proofs to students during lecture (e.g., Fukawa-Connelly, 2012; Mills, 2011; Weber, 2004). An important assumption behind this pedagogical practice is that mathematics majors can learn mathematics by reading and studying the proofs that their professors present. Unfortunately, research suggests that mathematics majors may learn little from studying these proofs. Cowen (1991) raised the issue of mathematics majors' inability to comprehend proofs to the mathematics community by noting:

> "If you need evidence that we have a problem, let one of your B students … explain the statement and proof of a theorem from a section in the book that you have skipped. My students, at least, do not have the innate ability to read and understand what they have read. When I ask them to read a problem and explain it to me, the majority just recite the same words back again" (p. 50).

This commentary is consistent with claims from mathematics educators that mathematics majors are confused by the proofs that they read (e.g., Conradie & Firth, 2000; Porteous, 1986; Rowland, 2001) Empirical findings also highlight mathematics majors' difficulties with proof reading by demonstrating that students often cannot distinguish an invalid argument from a valid proof (Alcock & Weber, 2005; Selden & Selden, 2003). Since much of advanced mathematical lectures involve the presentation of proofs and students in these classes often gain little from studying these proofs, it follows that students' learning in mathematics lectures is frequently rather limited.

Mathematics educators have suggested various causes for students' difficulties with proof comprehension, including students lacking the reading strategies necessary to understand a proof (e.g., Selden & Selden, 2003; Weber & Alcock, 2005), students gaining conviction from non-deductive arguments (Harel & Sowder, 1998), and students finding the format in which proofs are traditionally written to be confusing (Leron, 1983; Rowland, 2001). In this article, we argue that there is another important reason that mathematics majors have difficulty comprehending proofs: Mathematicians and mathematics majors have significantly different beliefs about what students' responsibilities are when they are asked to study a proof.

2. Related literature and theoretical perspective

2. 1. Students' beliefs about proofs

Most research into students' beliefs about proof can be classified into two categories. First, there is a large body of research into students' proof schemes, or the types of arguments that students find personally convincing or use to persuade others (Harel & Sowder, 1998). A general finding is that students often have "authoritative proof schemes", or believe a mathematical statement is true because an authoritative source told them that it was (e.g., Harel & Sowder, 1998). A student holding an authoritative proof scheme might not feel the need to follow the proof that their professor presented since they were convinced that the theorem was correct once the professor or textbook labeled it as a theorem. Other students gain conviction by verifying that a general mathematical claim is true with several specific examples (e.g., Healy & Hoyles, 2000; Recio & Godino, 2001; Segal, 2000). Such students might not find deductive proofs relevant as they seek mathematical conviction in a different way.

A second body of research into beliefs about proofs concerns one's perspective on the purpose of proof. In an influential article, de Villiers (1990) argued that mathematicians do not only produce proofs to become convinced that theorems are correct, but also to understand why theorems are true, systematize mathematical theories, discover new mathematics, and facilitate communication. Subsequently, mathematics educators have argued that proof should play a similar role in mathematics classrooms. In particular, mathematics educators have emphasized that the proofs that students study should promote understanding by explaining why theorems are true, rather than being a formal demonstration that is devoid of insight (e.g., Hanna, 1990; Hersh, 1993). Research has suggested that secondary students and secondary teachers do not share mathematics educators' perspectives on the purposes of proof; both populations tend to believe that the primary purpose of proof in the mathematics classroom is to provide conviction, while not appreciating the important roles of explanation or facilitating communication (e.g., Healy & Hoyles, 2000; Knuth, 2002). Whether mathematics majors feel the same way is an important open question.

2. 2. A reader's responsibility when reading a mathematical proof

In this paper, we focus on a different type of belief that a student might have with respect to proof. In our perspective, the act of proving a theorem can be understood as an interactional accomplishment between the prover and his or her audience (cf., Yackel & Cobb, 1994). To highlight the importance of audience when reading a proof, Manin (1977) noted "an argument becomes a proof after the social act of accepting it as a proof". Similarly, Devlin (2003) argued that instead of asking "what is a proof", it might be more germane to ask "when is a proof", emphasizing that an argument is not regarded as a proof the instant it is produced, but only is sanctioned as a proof after the mathematical community has scrutinized and critiqued it. These passages imply that for the mathematical community to sanction a theorem as proven, the prover and the larger community must each participate in the construction and validation of the argument.

The broad issue we address in this paper is how much of the responsibility in understanding a proof belongs to the prover and how much is left to the audience of the proof, particularly when the audience of the proof consists of students in a mathematics lecture. With regard to mathematicians' practice, much of the responsibility in the comprehension of a proof belongs to the reader of the proof. For instance, consider the excerpt below where one mathematician describes how he read a proof in a new mathematical domain:

> Mathematician: I'm doing a reading course with a student on wallpaper groups and there is a very elegant, short proof on the classification of wallpaper groups written by an English mathematician. So this is one where he's deliberately not drawing pictures because he wants the reader to draw pictures. And so I'm constantly writing in the margin, and trying to get the student to adopt the same pattern. Each assertion in the proof basically requires writing in the margin, or doing an extra verification, especially when an assertion is made that is not so obviously a direct consequence of a previous assertion.
>
> Int: So you're writing a lot of sub-proofs?

> Math: I write lots of sub-proofs. And also I try to check examples, especially if it's a field I'm not that familiar with, I try to check it against examples that I might know. (from Weber & Mejia-Ramos, 2011, p. 337).

In this excerpt, the mathematician views his responsibility when reading the proof to be significant. Many new assertions in the proof require the construction of a sub-proof and sometimes understanding them also necessitates the drawing of pictures or the consideration of examples. Indeed, this mathematician suggests that the author of the proof *deliberately* left the responsibility of drawing the appropriate pictures to the reader of the proof, presumably because the reader would gain understanding from engaging in this process.

2. 3. Reading a proof in the university mathematics classroom

We believe that a proof presentation might be regarded as successful in the advanced mathematics classroom if it enhances students' understanding of the mathematics being taught. Further, we adopt the position that a successful proof in this environment is an interactional accomplishment, where both the presenter of the proof (usually the professor) and the audience of the proof (the students) both need to do work to make sure this understanding is achieved. The central issue discussed in this paper is what each party feels their division of labor in this process is. How much work should the teacher do to make the proof comprehensible for his or her students? How much work are the students expected to do to resolve the difficulties they experience when reading a proof? If a student fails to understand a proof, is this because the quality of the presentation was inadequate or because the student did not exert sufficient effort to understand it?

Based on an interview study, Weber (2012) found that mathematicians expected undergraduate students to do substantial work to understand a proof. These mathematicians claimed that reading a proof was a complex and lengthy process, estimating that students should spend between 15 minutes and two hours studying proofs that were under ten lines long. Some mathematicians lamented that students did not appreciate how much responsibility they had when reading a proof, as the following quote illustrates.

> Mathematician: One of the things that students are *not at all* accustomed to is the notion that once you start reading mathematics, serious mathematics, you can't read it without a pencil in hand and that serious mathematics is encoded in a kind of short hand, and every sentence has to be decoded. (Weber, 2012, p. 475).

In this paper, we investigate what students believe is their responsibility when reading a mathematical proof and what mathematicians believe the students' responsibilities, highlighting discrepancies between the two.

3. Unproductive beliefs about proof reading: Hypotheses generated from a qualitative study on mathematics majors' proof reading

In this section, we present four unproductive beliefs that we hypothesized that students held based on our observations of how mathematics majors read proofs. These observations occurred in the context of a qualitative study in which 28 mathematics majors who had recently completed a transition-to-proof course were audiotaped while evaluating the validity of ten mathematical arguments. These students were all sophomores or juniors. They were each asked to read ten mathematical arguments and then asked to evaluate the argument on three criteria: Did they feel they understood the argument? Did they find the argument convincing? Did they think the argument constituted a mathematical proof? After evaluating each of the arguments, participants were asked questions about their proof reading processes and beliefs, such as "what do you think makes a good proof?"

The main results of this study—that is, how the participants evaluated the ten arguments—as well as a detailed description of the research methods, are presented elsewhere (Weber, 2010). In general, the participants did not perform well on the evaluation tasks, accepting the majority of invalid arguments as valid proofs. When we considered why participants had difficulty identifying which arguments were invalid, we noticed that the participants tended to read the arguments in a superficial manner, focusing largely on the calculations of what they read and lamenting that the author of the proof did not provide them with information that we felt the participants could have derived themselves. From this, we generated four hypotheses about unproductive beliefs that mathematics majors hold about proof reading. We describe how we came to conjecture these beliefs below. Although the results of mathematics majors' evaluations have been reported in another paper (Weber, 2010), the data presented in this section has not been presented elsewhere. At this stage, the evidence to support them has an anecdotal feel, but in the sections that follow, we describe a large-scale survey that we conducted to confirm these hypotheses.

Belief #1. Students should not be expected to justify statements within a proof. If a statement is given within a proof, then it should be explicitly stated how that assertion follows from previous assertions.

The 28 participants in the proof reading study were asked what they thought made a good mathematical argument. Sixteen of the 28 participants indicated that a good mathematical argument should explicitly list all the logical details and justifications within a proof. To illustrate, two excerpts of students' responses are given below:

> P9: It's got to be really detailed. You have to tell every detail. Every step, it is very clear. I like doing things step by step.
> Interviewer: So you like having every detail spelled out as much as possible?
> P9: Yeah, yeah, yeah.
>
> P10: Well, one it has to cover all the bases so that it is in fact a complete rigorous proof. For me, as a student, what else I would like to see is all the intermediate sorts of steps, things to help along, graphs, visual things. Things that recalled facts that perhaps I should know but you know, maybe not immediately at the tip of my tongue. That's to me what makes a good mathematical argument.

As noted previously, this viewpoint seems to be in contrast to mathematicians, who expect to spend a great deal of time constructing sub-proofs when they read a proof. If mathematicians assume their students will fill in the logical details of the proofs that they present but mathematics majors do not share this expectation, then it is likely that mathematical majors will not do the work required of them to understand a proof.

Belief #2: Understanding a proof consists entirely of understanding how each statement in the proof is justified.

The 28 participants were also asked what they do when they read proofs in their mathematics classes. Half of the participants indicated that they would simply check that every new statement in the proof followed from previous statements, as we illustrate below:

I: The professors in your classes, give a proof, what do you do?
P7: I copy that down and thinking of that when I copy that.
I: When you are copying it what are you thinking about?
P7: For if it makes sense, for every step.

I: Okay, if every step follows from the last one. Do you do anything differently when you study it at home?

P7: Study. When I read the book, read the proof, I do the same thing.

Several students said they would read proofs until they felt they understood them. When asked what it meant to understand a proof, these participants described understanding as being able to justify each step in the argument. Two representative responses are given below:

P10: To understand a mathematical argument? That you read it, that you understood it step by step, and certainly any facts that it appeals to, that you either know them at the top of your head or you've gone and looked them up and come to a point where you could, with very little prompting, reproduce it or certainly explained it to somebody else.

P11: To understand every logical step, and be able to know the reason why each step was done, and understand that it was a valid step, logically.

These responses are consistent with the participants' observed behavior when reading the arguments in our study; most participants focused almost entirely on justifying how new assertions in a proof followed from previous ones (a finding also reported in Inglis & Alcock, 2012, and Selden & Selden, 2003). Based on this, we hypothesized that mathematics majors might believe that understanding a proof consists entirely of being able to verify how each statement in a proof follows validly from previous assertions, although we concede that it is possible that the students in the studies cited above behaved in this way because they were validating the proofs rather than trying to comprehend them. This viewpoint is in contrast to mathematicians that we interviewed who claimed they presented proofs to students to explain why theorems were true and illustrate methods that might be useful for proving other theorems (Weber & Mejia-Ramos, 2011). If mathematics majors and mathematics professors disagree on what it means for students to understand a proof, then it is likely that mathematics majors are not reaping the insights that professors expect when the professors present proofs to them.

Belief #3: Mathematics majors do not think that studying a proof is a lengthy process.

In our proof reading study, participants rarely spent over five minutes reading a proof that they were given; in fact they often studied the argument for under two minutes before evaluating its validity and persuasiveness. This occurred even in cases where participants claimed that they did not fully understand the argument. Based on these observations, we hypothesized that students might not realize that studying a proof is a complicated and time-consuming process, although other alternative hypotheses can account for these observations as well (e.g., the participants lacked the motivation to study the proofs carefully or believed the goal of the study was to make quick evaluations). As we noted earlier, such a belief would be in contrast to mathematicians whom we interviewed, who expect students to spend 15 minutes to two hours studying the types of proofs that an undergraduate might typically read. We note here that Belief #3 might be a consequence of Belief #1 and Belief #2—if understanding a proof consisted of understanding the logical justifications inherent in the proof and all logical justifications should be explicitly provided for the reader, then it should not take long to develop a good understanding of the proof.

Belief #4. Students should not be expected to draw diagrams to help them understand a proof. If a diagram would aid a students' comprehension of a proof, then it should be provided within that proof.

One of the arguments that participants evaluated demonstrated that $4x^3 - x^4 = 30$ had no solutions. The argument consisted of using calculus to show the function $f(x)=4x^3 - x^4$

had a global maximum of 27 at $x = 3$; the argument relied on the fact that $f(x)$ must have a global maximum as $f(x)$ diverged to negative infinity as x approached both positive and negative infinity. Fifteen of the 28 participants who read this argument were confused by the fourth line of the proof, which asserted, "Because $f(x)$ is a polynomial of degree 4 and the coefficient of x^4 is negative, $f(x)$ is continuous and will approach $-\infty$ as x approaches ∞ or $-\infty$. Hence, $f(x)$ must have a global maximum." Of these 15 participants, seven were able to convince themselves that the fourth line of the proof was valid, with three of these seven participants drawing a graph. The other eight participants did not resolve their confusion. None of these participants attempted to draw a graph, even though four explicitly remarked that a graph would help them understand the fourth line of the proof. More generally, participants rarely drew graphs or diagrams to help them understand the arguments that they were reading; there were only four such instances of this observed in our study. Based on this, we hypothesized that mathematics majors may not view it as their responsibility to draw a graph as an aid for comprehension of a given proof (although again other explanations are possible, such as students lacking the content knowledge to draw a graph of a quartic equation with a negative lead coefficient). We note that several mathematicians whom we interviewed emphasized that students would need to draw diagrams to understand some of the proofs that they read (Weber, 2012).

4. Methods

4. 1. Rationale for this study.

In the previous section, we described four potentially unproductive beliefs about proof that we hypothesize that undergraduates hold. However, these hypotheses should be viewed very tentatively and speculatively as they were based on anecdotal evidence from a study regarding proof validation, not proof comprehension. For instance, the claim that mathematics majors do believe that proof understanding is a time consuming process was based on the fact that students in the validation study quickly read the arguments they were evaluating. While such an account is plausible, it is also conceivable that participants were not motivated to invest time studying these proofs as their course grade did not depend on their performance in the study. Similarly, the claim that mathematicians find these beliefs unproductive could use further support. In general, the evidence is based on the fact that in interviews with nine mathematicians from a single prestigious mathematics department about their professional and pedagogical practice with respect to proof, they claimed to hold the opposite beliefs (Weber, 2012; Weber & Mejia-Ramos, 2011). However, due to a relatively small and potentially biased sample of mathematicians, we cannot be certain that the viewpoint of the interviewed mathematicians is consistent with that of the broader mathematical community.

The goal of the current study was to attempt to document that (a) the four hypothesized beliefs we listed were held by the majority of mathematics majors, (b) the majority of mathematicians would not want their students to hold these beliefs, and (c) there is a statistically reliable difference between mathematicians' and mathematics majors' perceptions about these beliefs.

4. 2. The use of an internet-based study

Following the methodology employed by Inglis and Mejia-Ramos (2009), we collected data through the internet in order to maximize our sample size of mathematics majors and mathematicians. Recent studies have examined the validity of internet-based experiments by comparing a series of internet-based studies with their laboratory equivalents (e.g. Kranz & Dalal, 2000; Gosling et al., 2004). The notable degree of congruence between the two methodologies suggests that, by following simple guidelines (as listed in Inglis & Mejia-Ramos, 2009), internet data has comparable validity to more traditional data. Given these safeguards and the impracticality of obtaining a large sample of mathematicians in any other fashion, we believe the use of an internet study is justified.

4. 3. Participants.

To recruit participants, two e-mails were sent to the mathematics department secretary at a large state university in each of the 50 states in the United States[1]. In the first e-mail, the secretary was asked to forward a message to undergraduate mathematics majors inviting them to participate in a survey about how they read proofs in their mathematics classes if they had successfully completed a proof-oriented mathematics class, assuring them that their responses were completely anonymous, with a hyperlink for them to click on should they choose to participate. Upon clicking on that link, participants were asked for demographic information, including the questions, "Have you ever taken a mathematics class where you were expected to write proofs regularly?" If a participant answered no to this question, his or her responses were not included in the data to be analyzed. There were 175 participants who indicated that they had completed such a course and completed every survey item.

The second e-mail to the department secretary asked him or her to forward an e-mail to the mathematics department, inviting mathematicians who had taught a proof-

[1] In general, these universities were either named "[state name] University", "University of [state name]", or "[state name] State University". In states where multiple options were available and one of the universities had a reputation of having a prestigious mathematics department, we chose not to invite this university to ensure we were examining the beliefs of a typical, rather than "elite", mathematics major. If there were still multiple universities to choose from, we would choose "[state name] State University".

oriented course to participate in a survey on how they would like their students to behave in these courses. They were given a hyperlink to click on (that was different from the link for mathematics majors) if they were interested in participating. Upon clicking on that link, participants were asked for demographic information, including the questions, "Have you ever taught a mathematics course where your students were asked to read and write proofs regularly?" If a participant answered no to this question, his or her responses were not included in the data to be analyzed. There were 83 participants who indicated that they had taught such a course and completed every survey item. As participants were assured that their responses were completely anonymous, we do not know which universities had participants who chose to respond to our surveys.

4. 4. Materials and procedure.

For Beliefs #1 and #4, we created survey items for mathematics majors consisting of two contradictory beliefs A and B. Belief A is the belief we hypothesized that students held, while Belief B represented an opposite point of view. We strove to write the items so that they would both sound plausible to a mathematics major—that is, we tried to avoid a situation in which a student would choose one belief over another because the former sounded like the "right" belief. To illustrate, for Belief #1, that participants believed in a good proof, every step was explicitly justified, we created the following choices:

> A. In a good proof, every step is spelled out for the reader. The reader should not be left wondering where the new step in the proof comes from.
>
> B. When reading a good proof, I expect I will have to do some of the work to verify the steps in the proof myself.

Similarly, for Belief #4, the mathematics majors were shown the following choices:

A. When reading a good proof, if a diagram can help my understanding, it should be included. I should not be expected to draw a diagram myself.

B. When reading a good proof, sometimes diagrams are not included. I expect to have to sometimes have to draw these diagrams myself.

After mathematics majors were shown these two choices, they were asked to whether Statement A or Statement B was more indicative of their personal viewpoint using a five-point Likert scale, where their choices were "I strongly prefer statement A [statement B]", "I prefer statement A [statement B]", and "I am neutral between statements A and B".

Mathematicians were shown a pair of statements that were nearly identical to those shown to the mathematics majors. However, instead of directing the questions to their experience reading proofs, the questions were phrased about mathematics majors. For instance, for Belief #1, mathematicians saw the statements:

A. In a good proof, every step is spelled out for the student. A mathematics major should not be left wondering where a new step in the proof came from.

B. In a good proof, I expect that a mathematics major will still have to do some of the work to verify some of the steps in the proof.

They were then asked which statement was more indicative of their personal understanding using a five-point Likert scale.

For Belief #2, students were shown a single statement—"If I can say how each statement in a proof follows logically from previous statements, then I understand the proof completely" and asked to judge whether they disagreed with statement using a five-point Likert scale. For Belief #3, mathematics majors were asked to indicate how long they should typically spend studying a proof that was presented to them in class using an open response box. (Note this asked how long they *should* read a proof, not how long they *do* read a proof, as we wanted to focus on their beliefs rather than on their behavior). Again, for Belief #2 and Belief #3, mathematicians were asked analogous questions about mathematics majors' proof reading rather than their own proof reading. The exact questions that mathematics majors were asked are given in the forthcoming Results section. The specific questions that the mathematicians were asked are given in the Appendix.

4. 5. Analysis

For Belief #3, a t-test was used to compare whether mathematics majors expected to spend less time studying a proof than mathematicians expected them to. For each of the remaining beliefs, three statistical tests were performed. For both the mathematicians and the mathematics majors, we conducted one-sampled Wilcoxon ranked sign tests to see if the mathematics majors' and the mathematicians' median responses differed reliably from neutral. We also conducted Mann-Whitney tests comparing mathematics majors' and mathematicians' responses to the analogous items to see if there was a difference in mathematicians and mathematics majors about students' responsibility when reading proofs.

5. Results

A summary of participants' response to the survey is presented in Table 1.

Table 1.
Mathematics majors' and mathematicians' responses to the survey items

		Prefer Statement A	Neutral	Prefer Statement B
Belief #1	Math majors	75%*	11%	14%
	Mathematicians**	27%	21%	52%*
Belief #2	Math majors	75%*	12%	13%
	Mathematicians**	23%	10%	67%*
Belief #4	Math majors	66%*	15%	19%
	Mathematicians**	32%	19%	49%

Note for Belief #2, participants were asked whether they agreed or disagreed with the hypothesized unproductive belief. In these cases, "Preferred Statement A" indicated participants agreed with the statement and "Preferred Statement B" indicated that participants disagreed with the statement.

*- Indicates that a one-sampled Wilcoxon ranked sign test indicated that the group's median response differed significantly from neutral.
**- Indicates a Mann-Whitney test indicates that mathematicians and mathematics' majors to these questions differed significantly.

Belief #1.

Mathematics majors were asked which of two statements they preferred:

A. In a good proof, every step is spelled out for the reader. The reader should not be left wondering where the new step in the proof comes from.

B. When reading a good proof, I expect I will have to some of the work to verify the steps in the proof myself.

The large majority of mathematics majors (75%) preferred statement A, with 47% of the mathematics majors strongly supporting statement A. The majority of mathematicians (52%) expected that mathematics majors would have to do some of the work in verifying the steps in the proof. Mathematics majors were more likely than mathematicians to support statement A (U=3128.5, p<.001). This confirms our hypothesis that most mathematics majors do not believe it is their responsibility to verify assertions while reading a proof. However, although relatively few mathematicians (23%) agreed with statement A, we were surprised that only 52% of the mathematicians agreed with statement B. In the Discussion section, we address the variance in mathematicians' responses.

Belief #2.

Mathematics majors were asked if they agreed or disagreed with the statement:

If I can say how each statement in a proof follows logically from previous statements, then I understand the proof completely.

The majority (75%) of mathematics majors agreed with this statement while the majority of mathematicians (67%) disagreed with this statement. Mathematics majors were more likely than mathematicians to support this statement (U = 2642, p<.001).

Belief #3.

Mathematics majors were asked "how long should you typically spend reading a proof that is presented to you in your classes (in minutes)?" Some participants' responses were not numerical (e.g., "as long as it takes") and we did not include these responses in our analysis. While some participants contained a specific time (e.g., "15 minutes"), others contained a range of times (e.g., "10 to 15 minutes" or "one to two hours"). To analyze the data, for each participant who entered numerical data, we coded them as having a minimum time and a maximum time. For instance, if a participant said, "10 to 15 minutes", we coded the minimum time as 10 minutes and the maximum time as 15 minutes. If the participant gave an exact time, we included that time as both his or her minimum and maximum time.

The 156 mathematics majors who provided numerical data had a minimum mean time of 17 minutes and a maximum mean time of 20 minutes. The 52 mathematicians who provided numerical data has a minimum mean time of 30 minutes and a maximum mean time of 37 minutes. Both the minimum mean time (t(206) = 3.346, p=.001) and the maximum mean time (t(206) = 5.093, p<.001) were greater for mathematicians than they were for mathematics majors. For the maximum time, 41% of the mathematics majors gave a time of greater than 15 minutes or more, indicating the majority of mathematics majors believed it would be unusual for them to have to spend more than 15 minutes studying a proof. In contrast, 81% of mathematicians listed a maximum time of greater than 15 minutes.

Belief #4.

Mathematics majors were asked which of the following two statements they preferred:

> A. When reading a good proof, if a diagram can help my understanding, it should be included. I should not be expected to draw a diagram myself.
>
> B. When reading a good proof, sometimes diagrams are not included. I expect to have to sometimes have to draw these diagrams myself.

The majority (66%) of mathematics majors agreed with statement A and mathematics majors showed a stronger preference for statement A than mathematicians (U=4404.5, p<.001). However, it was again surprising that only 49% of mathematicians preferred statement B, a point that will be addressed in the discussion section. Nonetheless, if *some* mathematicians expect students to draw diagrams to understand some proofs that they read and *most* mathematics majors do not believe they should have to do so, then this can be a significant barrier to their comprehension of proofs.

5. Discussion

5.1. Summary of the results.

The results of this study provide qualitative and quantitative evidence that mathematics majors and mathematicians have different beliefs about the role students should take when reading proofs. Most mathematicians expect that mathematics majors will need to spend more than 15 minutes studying some of the proofs that are presented to them, but the majority of mathematics majors do not expect that they will need to do this. Most mathematics majors believe they understand a proof completely if they can justify each step in the proof, while mathematicians generally believe understanding a proof consists of more than this. Mathematics majors indicate they think that reading a good proof should be somewhat of a passive process in the sense that they do not believe they will have to construct sub-proofs or diagrams if the proof was written well. However, many mathematicians expressed the opposite point of view.

5. 2. Implications for teaching and research on proof reading and presentation

The results from this paper offer an explanation for why mathematics majors learn little from the proofs that they read. Mathematics majors are likely not putting in the work that their professors expect. Many probably are not engaging in the activities required to comprehend a proof, such as constructing sub-proofs, drawing diagrams, or trying to understand a proof not merely as a chain of inferences, but in terms of its overarching methods or structure. However this may not be due to indolence on the part of the students. Mathematics majors may not engage in these behaviors because they are not aware of the benefits of doing so.

To help mathematics majors understand proofs, some mathematics educators have suggested presenting proofs in alternative formats that they believe will increase comprehension (e.g., Alcock, & Inglis, 2010; Leron, 1983; Rowland, 2001). Studies on the efficacy of these pedagogical suggestions are uncommon, but the quantitative studies that do exist have found little benefits in comprehension for these novel formats (e.g., Fuller et al, 2011; Roy et al, 2010; Weber et al, 2012). Our data suggest a reason why these alternative formats might not have been successful in these studies. Regardless of how proofs are presented, the reader still needs to actively interact with the proof if comprehension is to occur, and this likely includes constructing sub-proofs and understanding the proof as more than a series of inferences. As most mathematics majors do not accept this viewpoint, it seems they may not understand the proofs that they read even if they are presented differently. Indeed, Roy, Alcock, and Inglis (2010) suggested that making aspects of a proof more transparent may limit comprehension because this will encourage some students to take a less active role when reading the proof. We suggest that improving students' comprehension of proofs will require refining their perceptions of their responsibility while reading proofs.

5. 3. Variance in mathematicians' responses about justifications and diagrams

Although mathematicians were more likely than mathematics majors to agree with the view that students would sometimes have to construct justifications and diagrams when reading good proofs, their level of agreement was less than we predicted. In a previous study that we conducted, we observed a similar finding. We asked 110 mathematicians if adding a particular algebraic step in a proof would improve or lessen its pedagogical quality; 41 thought the addition made the proof pedagogically better, 40 thought it made the proof worse, and 29 thought it did not affect the pedagogical quality of the proof. Participants who thought the addition made the proof better emphasized that one should be as clear as possible when writing proofs while those who thought the addition made the proof worse said the step should be obvious to students or students would benefit from filling in these types of gaps (Lai, Weber, & Mejia-Ramos, 2012). Hence mathematicians might disagree on what types of inferences students are capable of making. Findings of this type suggest that mathematicians might not be uniform in how much detail they provide in their proofs and what they expect their students to do when reading proofs. As mathematics professors in general seem to spend little time telling students how to read proofs (cf., Weber, 2012), it seems likely that students are not made aware of these differences, which understandably could lead them to be confused about their role as a proof reader.

5. 4. How are these unproductive beliefs acquired?

We suggest that the unproductive beliefs that students hold about proof reading could be shaped by their early experiences learning proof and the different roles that proof plays in mathematics classrooms. Regarding the first point, students are usually first exposed to proof in high school geometry. In this environment, proofs are typically written in a two-column format (Herbst, 2002) where each step in the proof is explicitly justified with a geometric principle and each theorem is usually accompanied by a diagram. The emphasis in presenting these proofs and in assessing student contribution typically is on form and correctness rather than on meaning (Schoenfeld, 1988). Transition-to-proof courses are not quite as rigorous, but much time in these courses is spent deducing obvious results (e.g., the product of two odd numbers is odd) and each step in a proof is expected to be justified, even in cases where the mathematics professors would be capable of supplying the justification themselves with minimal effort (e.g., Weber, 2008). It seems natural that students might generalize from these experiences and believe that every step in a good proof should be explicitly justified and that they would think understanding a proof was tantamount to being able to justify every step within the proof. We also note that in advanced mathematics courses, proofs are usually presented relatively quickly and students' comprehension of a proof is rarely assessed (cf., Mejia-Ramos et al, 2011; Weber, 2012). Consequently we should not be surprised that students do not think they are expected to spend a great deal of time studying the proofs that are presented to them.

At a broader level, we contend that the proofs that students hand in for credit and the proofs that professors present to them serve different epistemic purposes. The proofs that professors present to students typically have communicative purposes. While these proofs provide explanation as to why theorems are true and reveal new techniques for solving mathematical problems, the proofs that students produce in these classes usually do not have communicative purposes. Professors rarely read the proofs that students submit in search of insight. Also, in most classes, students spend little time reading each other's proofs. Instead proving tasks are given by professors as a pedagogical tool to reveal how well the students know the material that is being studied or how well they could construct proofs themselves. For the purposes of revealing student reasoning, more detail is desirable in that it provides more insight into what the students actually know. In contrast, for the purposes of communication, providing less logical details in a proof and focusing on the bigger ideas in the proof is often advantageous. Not only is an excessively detailed proof time-consuming to present, but the larger important conceptual ideas in the proof might be masked by more mundane logical manipulations. Mathematics majors may be unaware both that the standards to which their proofs are held differ from the standards of the proof presented to them in class and why this is so. Consequently, instruction may be improved, both by making students aware of these distinctions and with giving students the opportunity to use proof as a communicative tool in meaningful mathematical activity rather than merely as an activity used to assign course grades.

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Appendix- Survey items seen by mathematicians

Belief #1:
Consider the following statements:
A. In a good proof, every step is spelled out for the student. A mathematics major should not be left wondering where a new step in the proof came from.
B. In a good proof, I expect that a mathematics major will still have to do some work to have to verify some of the steps in the proof.
Indicate whether statement A or statement B is more indicative of your personal viewpoint:

Belief #2:
Consider the following statement:
If a mathematics major can say how each statement in a proof follows logically from previous statements, then that student understands this proof completely.
Which of the following best reflects your views on this statement? [Participants are then asked whether they agree or disagree with the statement using a five-point Likert scale].

Belief #3:
Ideally, when mathematics majors are given a theorem and a proof, how long would you like these mathematics majors to spend considering the theorem statement prior to reading its proof (in minutes)?

Belief #4:
Consider the following statements:
A. In a good proof, if a diagram can help a mathematics majors' understanding, it should be included with the proof. Mathematics majors should not be expected to draw diagrams when reading proofs.
B. In a good proof, sometimes useful diagrams are not included. Mathematics majors will sometimes have to draw these diagrams themselves.
Indicate whether statement A or statement B is more indicative of your personal viewpoint: