INVERSE FUNCTIONS

Select one person to be the Teacher, and one person to be the Student. Re-enact the following scenario, and consider how you would respond as the teacher.

Classroom Scenario: A secondary teacher is discussing how to solve trigonometric equations.

Q1. Evaluate the student’s work – is it correct or incorrect? Provide feedback to the student and the class if there is anything incorrect.
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**Q2.** Suppose $f$ and $g$ are inverse functions. What does this mean to you? Consider: Are $\sin(x)$ and $\arcsin(x)$ inverse functions? (Note that $\arcsin(x)$ is the same as $\sin^{-1}(x)$) Explain.
**Definition 1.5.1a.** A function \( f : A \to \mathbb{R} \) is one-to-one (injective) if for each \( y \in f(A) \), there is a unique \( x \in A \) such that \( f(x) = y \). (i.e., if \( x_1 \neq x_2 \) in \( A \), then \( f(x_1) \neq f(x_2) \)).

**Definition 1.5.1b.** A function \( f : A \to B \) is onto (surjective) if \( f(A) = B \). (i.e., for any \( b \in B \), there is an \( a \in A \) such that \( f(a) = b \)).

**Definition 4.6.1** (adapted for “strictly”). A function \( f : A \to \mathbb{R} \) is strictly increasing on \( A \) if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) (and \( x_1, x_2 \in A \)) and strictly decreasing on \( A \) if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \) (and \( x_1, x_2 \in A \)). A strictly monotone function is one that is either strictly increasing or strictly decreasing.

**Definition** (Exercise 4.5.8). For a function \( f : A \to \mathbb{R} \) that is one-to-one on \( A \), the inverse function \( f^{-1} : f(A) \to \mathbb{R} \) is given by \( f^{-1}(y) = x \) where \( y = f(x) \). (The inverse function is characterized by the following relations: i) \( f^{-1}(f(x)) = x \) for all \( x \in A \); and ii) \( f(f^{-1}(y)) = y \) for all \( y \in f(A) \).
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**Q3.** Fill in the following statements with: always, sometimes, or never. Draw any examples or counterexamples that you consider. (Suppose $f$ is a function.)

i. If $f$ is strictly monotonic on an interval $I$ (and defined on every point of $I$), it will __________ have an inverse function.

ii. If $f$ is not monotonic on an interval $I$ (and defined on every point of $I$), it will __________ have an inverse function.

iii. If $f$ is continuous on an interval $I$ (and defined on every point of $I$), it will __________ have an inverse function.

iv. If $f$ is continuous and strictly monotonic on an interval $I$ (and defined on every point of $I$), it will __________ have an inverse function.

v. If $f$ is continuous and not strictly monotonic on an interval $I$ (and defined on every point of $I$), it will __________ have an inverse function.
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If we want an inverse for a continuous real-valued function $f$ but $f$ is not one-to-one, by convention, we seek to find the largest interval $A$ on which $f$ is monotonic such that $A$ contains 0 and at least one positive number.

**Q4.** Sketch the graph of $f(x) = \sin(x)$ and $g(x) = \arcsin(x)$. In what sense are $f(x)$ and $g(x)$ inverse functions? Is it always the case that $f(x) = y$ implies that $g(y) = x$.

**Q5.** Consider the graph of $h(x)$ below.

(a) Specify (by drawing) the interval $A$ that represents the conventional domain on which $h(x)$ has an inverse on the graph above.
(b) Justify your choice of this interval using the theorems that were previously presented.
(c) Would $h(x)$ have an inverse on the following domains? Again, justify your answer using the theorems that were previously presented.

(i) $\left[0, \frac{\pi}{4}\right]$
(ii) $\left[\frac{\pi}{2}, \pi\right]$
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Q6. Consider again the equation \( \sin(2x) = 0.7 \)

i. If \( f(x) = \sin(2x) \), from the graph of \( f(x) \), how can we restrict the domain of \( f(x) \) so that \( f(x) \) has an inverse?

ii. What is the solution to \( f(x) = 0.7 \) on this domain?

iii. How can we use the periodicity of \( f(x) \) to find some of the remaining solutions?

iv. How can we use the symmetry and periodicity of \( f(x) \) to find the remaining solutions?

v. On the graph below, we labeled eight different solutions \( A, B, C, D, E, F, G, \) and \( H \). Which solutions were found in step (ii) above? Which solutions were found in step (iii)? Which solutions were found in step (iv)?
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Additional HW exercises

AE.1. In what sense is \( g(x) = \sqrt{x} \) an inverse function to \( f(x) = x^2 \)? Write how you would explain this to a class of students?

AE.2. Consider the following typical solution and explanation to a simple quadratic equation that algebra teachers frequently provide to their students.

\[
x^2 = 25 \\
x = \pm 5
\]

“We solve this equation by taking the square root of both sides. \( \sqrt{25} \) is + and \( -5 \).”

This explanation is mathematically incorrect. By definition, the square root function, \( \sqrt{x} \), is the unique non-negative number whose square is \( x \). Thus, \( \sqrt{25} = 5, \sqrt{25} \neq -5 \). Justify the step in the equation using the ideas in class, particularly viewing \( g(x) = \sqrt{x} \) as the inverse of \( f(x) = x^2 \) with a restricted domain and using the symmetry of \( f(x) = x^2 \) to find the missing solution.

AE.3. One way to explain how to solve the equation \((x + 1)^2 = 9\), is to rely on something students generally know, which is that there are two possible real numbers, that when squared, yield 9. These are 3 and -3. So to solve the equation, the next step is: \( x + 1 = 3 \), and \( x + 1 = -3 \). Another way to explain is to say that whenever you introduce a square root on one side, you need to put a \( \pm \) on the other. So to solve the equation, the next step is \( \sqrt{(x + 1)^2} = \pm \sqrt{9} \), so \( x + 1 = \pm 3 \). Some students like “doing something – the same thing – to both sides” in each step, and so option 1 feels a little less systematic, and option 2 feels a little odd (you do the \( \pm \) only on one side...) A third way is to consider applying the inverse function of \( x^2 \), which can be considered on the restricted Domain \([0, \infty)\), and is \( \sqrt{x} \). When applying the inverse function to both sides, we get \( \sqrt{(x + 1)^2} = \sqrt{9} \), so \( x + 1 = 3 \). If you used this third approach, how would you help the student interpret the one solution they found, \( x = 2 \), and how would you help them find the remaining solution?