For each ULTRA Lesson, we “sandwich” some piece of real analysis content (e.g., definition, theorem) in between a connection to teaching secondary mathematics. This includes three phases: i) **BUILDING UP** – beginning with a discussion of a classroom situation and/or problematizing some of the (secondary) mathematics; ii) **REAL ANALYSIS** – teaching the real analysis content; and iii) **STEPPING DOWN** – highlighting how the real analysis content can be related to teaching and/or re-considering some of the (secondary) mathematics.

In general, the ‘building up’ portion of each lesson is intended to be done in small groups during class, as an introduction to and motivation for the real analysis content; that is, tasks/questions are written for students. The ‘real analysis’ content is presumed to be instructor-led; that is, it is written in a format primarily intended to be presented to students. Last, the ‘stepping down’ portions include both in-class tasks in small groups and HW exercises for individuals, with small group discussion of HW exercises to be conducted during a subsequent class.

**Module 5**

**REAL ANALYSIS Content**

**Definition 4.3.1.** Let $f : A \to \mathbb{R}$, and let $c$ be a point of the domain $A$. We say that $f$ is **continuous** at $c$ provided that for all $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - c| < \delta$ (and $x \in A$) then $|f(x) - f(c)| < \varepsilon$.

**SECONDARY Mathematics**

State the implications of choosing definitions on determining the types of objects that would fulfill such definitions (e.g., continuity, trapezoid, etc.).

**PEDAGOGICAL AIM – Secondary teaching practice**

**Principle of Teaching 1.** Be purposeful about the definitions, arguments, or explanations used with students, acknowledging and revisiting mathematical limitations – in this case, of recognizing that definitions have consequences for what is included in that set of objects and what statements we can make about them.

**Principle of Teaching 4.** Select examples that exemplify nuances within and boundaries around a mathematical idea – in this case, definitions.
Overview of Module 5

Driving mathematical question: Are objects defined or do we define them?
Most people tend to view mathematics as an established field of facts. However, what many people do not recognize is that the historical development of mathematics has always involved a process of debate – about what to study, how to study it, etc. It is true that much of the mathematics studied in schools has come to a point where there is general consensus about the objects being studied. But that has not always been the case. Consider, for example, the major ramifications of and debates between Cantor and Kronecker about defining infinite cardinalities in terms of bijective functions. Or Euclid’s preference for an exclusive definition of an isosceles triangle (i.e., excluding equilateral triangles) versus the common practice now of an inclusive definition. The point is, although in school mathematics there is general consensus about nearly all of the objects we study, these objects were not intrinsically defined for us – rather, we have created definitions and, throughout history, tended toward definitions that were particularly productive for one reason or another.

Driving pedagogical question: How can we use examples to portray definitions as having influence on the objects classified and the truth-value of statements made about them?
In mathematics, a normative approach to reading a definition is to understand the objects that would be included in this defined set. Prototypical examples are particularly useful for this. But so is thinking about potentially unusual ones. Doing so is an important component of understanding definitions as determining a class of objects, and, consequently, as understanding statements about those objects as applying to all such objects in that set. Instilling these sorts of conversations is productive not just for understanding a definition, but for actively engaging in the process of defining. That is, when asked to define an object, it is precisely these sorts of activities that are useful for both refining a definition and understanding the consequences of statements about defined objects. The pedagogical purpose of this module is to help PISTs recognize the importance of definitions in their own teaching, and how the way(s) in which they define objects can be consequential.
DEFINITIONS

In the last class, we said that a trapezoid is a quadrilateral with exactly one pair of parallel sides.

New York State, for their NCLB exams, has decided that they will define a trapezoid as a quadrilateral with at least one pair of parallel sides. (See below.) Mathematically, the definition of a trapezoid is not settled – there are a lot of people who use each definition.

New York State Common Core Geometry Standards Clarifications

In January 2011, the NYS Board of Regents adopted the NYS P-12 Common Core Learning Standards (CCLS), which include the Common Core State Standards and a small number of additional unique standards added by New York State. The CCLS were created through a collaborative effort on behalf of the National Governor’s Association Center for Best Practices and the Council of Chief State School Officers. The standards were developed by key stakeholders in the field, including teachers, school administrators, and content experts.

The main design principles in the NYS CCLS for Mathematics standards are focus, coherence, and rigor. These principles require that, at each grade level, students and teachers focus their time and energy on fewer topics, in order to form deeper understandings, gain greater skill and fluency, and more robustly apply what is learned.

In an effort to ensure that the standards can be interpreted by teachers and used effectively to inform classroom instruction, several standards of the Geometry curriculum have been identified as needing some clarification. These clarifications are outlined below.

Note: It is anticipated that more standard clarifications may be added to the list as feedback and requests for additional guidance are received.

Clarifications

G-CO.3
Trapezoid is defined as “A quadrilateral with at least one pair of parallel sides.”

Q1. If one classroom teacher used the “at least one pair of parallel sides” definition, and another used the “exactly one pair of parallel sides” definition, how would you help a student understand the difference between these two definitions? Explain your reasoning.
DEFINITIONS

Q2. List one (or more) definitions that you know for function continuity. That is, “A function is continuous if…”

Q3. Draw some i) examples, and ii) non-examples, of continuous functions below.
Definition 1.2.3. Given two sets $A$ and $B$, a function from $A$ to $B$ is a rule (or mapping) that takes each element $x$ in $A$ and associates with it a single element on $B$. In this case, we write $f: A \to B$.

Examples. Consider the following examples – are they functions?

i) $1.5x[[0.1x + 3]] - 1$ (zoomed in)
ii) $[[x]]$
iii) $1/x$
iv) $1/x$ for $x \neq 0$, $0$ for $x = 0$
v) $\sin(1/x)$
vi) $\sin(1/x)$ for $x \neq 0$, $0$ for $x = 0$
vii) Dirichlet function: $x$ if $x$ is rational, $0$ if $x$ is irrational
viii) $1/2|x - 1| + 1$ for $x \geq 0$, and $1$ for $x = -2$ (isolated point)
ix) Fourier series

Statement of purpose: We are having PISTs engage in this instructor-led activity to show that definitions matter, and a definition of a concept may imply that some surprising objects are members of that concept, even if they do not intuitively appear to be members of that concept.

Examples. Consider the following functions – which of these functions would be continuous according to the various definitions ((a) You can draw the function without picking up the pencil; and (b) $\lim_{x \to c} f(x) = f(c)$)?

i) $1.5x[[0.1x + 3]] - 1$ (zoomed in)
ii) $[[x]]$
iii) $1/x$
iv) $1/x$ for $x \neq 0$, $0$ for $x = 0$
v) $\sin(1/x)$
vi) $\sin(1/x)$ for $x \neq 0$, $0$ for $x = 0$
vii) Dirichlet function: $x$ if $x$ is rational, $0$ if $x$ is irrational
viii) $1/2|x - 1| + 1$ for $x \geq 0$, and $1$ for $x = -2$ (isolated point)
ix) Fourier series

Statement of purpose: Whether or not a mathematical object has some property depends on the definition – and not all definitions agree on every case. Knowing and exemplifying objects and non-objects for various definitions is important.

Definition 4.3.2. (c) Let $f: A \to \mathbb{R}$, and let $c$ be a point of the domain $A$. We say that $f$ is continuous at $c$ provided that for any sequence $(x_n)$ that has a limit of $c$ (with all $x_n$ in $A$), the limit of the sequence $f(x_n)$ is $f(c)$.

Definition 4.3.1. (d) Let $f: A \to \mathbb{R}$, and let $c$ be a point of the domain $A$. We say that $f$ is continuous at $c$ provided that for all $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - c| < \delta$ (and $x \in A$) then $|f(x) - f(c)| < \varepsilon$.

Examples. Consider the following functions – which of these functions would be continuous according to the various definitions ((c) and (d))?
i) \(1.5x[[0.1x + 3]] - 1\) (zoomed in)

ii) \([x]\)

iii) \(1/x\)

iv) \(1/x\) for \(x \neq 0\), 0 for \(x = 0\)

v) \(\sin(1/x)\)

vi) \(\sin(1/x)\) for \(x \neq 0\), 0 for \(x = 0\)

vii) Dirichlet function: \(x\) if \(x\) is rational, 0 if \(x\) is irrational

viii) \(1/2|x - 1| + 1\) for \(x \geq 0\), and 1 for \(x = -2\) (isolated point)

ix) Fourier series

**Statement of purpose:** The point is to further explore the influence of definitions by the standard definitions of continuity in real analysis. Further, even though (c) and (d) look different, they actually are equivalent definitions and hence admit the same objects.

**Statement 4.3.2.** Definitions (c) and (d) are equivalent definitions for continuity; when \(c\) is a limit point of the domain, definitions (b) and (d) are also equivalent.

**Definition 4.2.1.** Let \(f: A \to \mathbb{R}\), and let \(c\) be a limit point of the domain \(A\). We say that \(\lim_{x \to c} f(x) = L\) provided that for all \(\epsilon > 0\), there exists a \(\delta > 0\), such that if \(0 < |x - c| < \delta\) (and \(x \in A\)) then \(|f(x) - L| < \epsilon\).

**Statement of purpose:** The purpose here is now to sanction the definition of a key concept that will be used throughout the semester.
DEFINITIONS

Q4. The definition of isosceles trapezoid that each of the teachers used was “A trapezoid is isosceles if the non-parallel sides are congruent.” What are some implications of keeping this definition? Does this definition still make sense if we use the new, inclusive, definition of trapezoid? Why or why not?

Q5. How would your analysis of each of the teacher’s statements change had they been using the “at least one pair of parallel sides” definition? Would you have Teacher A or B respond differently to the student? Explain.
DEFINITIONS
Additional HW exercises

AE.1. Zero is an even number. However, students often suggest that zero is neither even nor odd. Which of the following would still be true if all other integers (positive and negative) except zero retain their even or odd status. Justify your response for each statement.

- even + even = even
- odd + odd = even
- even + odd = odd
- even * even = even
- odd * odd = odd
- even * odd = even

AE.2. If a Calculus student thought about continuity as whether or not the function could be drawn without picking up a pencil, how might you push their thinking so that if they were asked on the AP Calculus exam whether \( f(x) = \frac{1}{x} \) was continuous on its domain, s/he would give the correct answer?