For each ULTRA Lesson, we “sandwich” some piece of real analysis content (e.g., definition, theorem) in between a connection to teaching secondary mathematics. This includes three phases: i) BUILDING UP – beginning with a discussion of a classroom situation and/or problematizing some of the (secondary) mathematics; ii) REAL ANALYSIS – teaching the real analysis content; and iii) STEPPING DOWN – highlighting how the real analysis content can be related to teaching and/or re-considering some of the (secondary) mathematics.

In general, the ‘building up’ portion of each lesson is intended to be done in small groups during class, as an introduction to and motivation for the real analysis content; that is, tasks/questions are written for students. The ‘real analysis’ content is presumed to be instructor-led; that is, it is written in a format primarily intended to be presented to students. Last, the ‘stepping down’ portions include both in-class tasks in small groups and HW exercises for individuals, with small group discussion of HW exercises to be conducted during a subsequent class.

### Module 4

**REAL ANALYSIS Content**

**Theorem 2.4.2** (Monotone Convergence Theorem). If a sequence is monotone and bounded, then it converges.

- **Corollary** (Divergence Description). If a sequence diverges, then it is either not monotone or unbounded.

**Theorem 2.5.2.** Subsequences of a convergent sequence converge to the same limit as the original sequence.

- **Corollary** (Divergence Criterion). If there exists two subsequences of a sequence that converge to two different limits, then the sequence diverges.

**SECONDARY Mathematics**

Translate various statements about secondary mathematics into their logical form, differentiating between an implication and a bi-implication, converses and contrapositives, negations, and quantifiers.

**PEDAGOGICAL AIM – Secondary teaching practice**

**Principle of Teaching 5.** Clarify the logic underlying students’ mathematical statements, arguments, proofs or processes.
Overview of Module 4

Driving mathematical question: What is the logical structure of everyday statements?
Mathematical concepts and propositions are typically defined formally. However, good teachers do not convey the meaning of these ideas with strictly formal language. They do so by translating these formal ideas into everyday language that students can understand. However, this translation can be problematic, especially because words like “and”, “or”, “if”, and “implies”, have different meanings in formal logic and everyday language. Yet teachers and students alike often summarize these statements using these terms, which can mask the meaning of logical ideas or make the statements mathematically inaccurate. Being able to state mathematical ideas in ways that are comprehensible but precise, as well as recognizing the logical structure of informal statements, are important facets of mathematical communication.

Driving pedagogical question: How can attending to the logical structure of statements inform a teacher’s understanding of and response to students?
Many people underestimate the complexity of trying to specify the logical structure of everyday statements. Indeed, in the classroom, sources of miscommunication can go undetected because they relate to misunderstandings about or non-recognition of the logical underpinnings of a statement. For example, “is” in one context might indicate an implication and in another context a definition (biimplication); the parts of a sentence intended to be included in statements with “and” or “or” can be unclear; the order of mathematical quantifiers matters, etc. The pedagogical aim of this module is twofold. First, the goal is to convince teachers that paying attention to the logical structure of statements is an important component of mathematical communication. Second, the goal is to provide teachers with some experience trying to translate everyday statements into their logical form, and to recognize the potential benefits of doing so – for example, it might make writing proofs a little easier, or creating a logically equivalent statement (e.g., contrapositive) might communicate some other consequences more clearly.
Classroom Scenario: Below are two Geometry teachers teaching about isosceles trapezoids, Class A is the left column while Class B is the right column.

As a reminder, the definition of a trapezoid is a quadrilateral with exactly one pair of parallel sides.

Q1. How would you respond to the student?
**LOGIC**

*Definition (on board):* A trapezoid is isosceles if the non-parallel sides are congruent.

**Statement 1:** We can also see that the diagonals are congruent.

**Q2.** Write Statement 1 (made by both Teacher A and B) as a logically equivalent if-then statement.

**Q3.** Write each of Teacher A and Teacher B’s Statement 2 (shown below) as a logically equivalent if-then statement:

- **Teacher A Statement 2:** This is the same as saying a trapezoid is isosceles if the diagonals are congruent.
- **Teacher B Statement 2:** This is the same as saying that an isosceles trapezoid is a quadrilateral with congruent diagonals.

**Q4.** Which, if either, of Teacher A’s and Teacher B’s Statement 2 can be interpreted as an if-and-only-if statement based on the teacher’s phrasing? (Do not worry about whether the statement would be true, just how it is phrased.) Show and explain.

**Q5.** The student’s comment is “But wait, I don’t think that’s true, a rectangle has congruent diagonals.” In light of each of Teacher A and Teacher B’s Statement 2, which, if either, does the student’s comment respond to (do not worry about whether they are true, just how they are phrased)? Why?
LOGIC

Q6. For each of the following two statements, rephrase as an equivalent “if-then” or “if and only if” statement. (Do not worry, for now, about whether they are true.)

i. Sequences that are monotone and bounded, converge.

ii. Subsequences of a convergent sequence converge to the same limit as the original sequence.

Q7. Determine whether you think the statements you just wrote are true or false. Explain your reasoning.

Q8. Would any of the statements above tell us anything about divergent sequences? If so, which statement(s), and what exactly would it say about divergent sequences?
Theorem 2.4.2 (Monotone Convergence Theorem). If a sequence is monotone and bounded, then it converges.

Given: $a_n$ is monotonic – without loss of generality, assume $a_n$ is increasing $a_n$ is bounded: i) $\exists M$ s. t. $|a_n| \leq M$ for all $n$; ii) if $A = \{a_n|n \in \mathbb{N}\}$, then sup $A$ exists

To Prove: $\forall \varepsilon > 0, \exists N$ s. t. $\forall n \geq N, |a_n - a| < \varepsilon$

Proof. Let $\varepsilon > 0$. Let $a = \sup A$. Since $a = \sup A$, $a - \varepsilon$ cannot be an upper bound. Therefore, there exists an element of $A$, $a_N$, such that $a - \varepsilon < a_N$.

Since $a_n$ is increasing, $a_n \leq a_{n+1}$ for all $n$. Therefore, for all $n \geq N$, then $a_N \leq a_n$. Select $N$ such that $a - \varepsilon < a_N$. Then $\forall n \geq N, |a_n - a| < \varepsilon$. QED.

Statement of purpose. This is an important theorem in real analysis. However, we also use this as an opportunity to illustrate how logical statements can be unpacked. This relates to the mathematical work that PISTs engaged in with Q5. In particular, we unpack a complex hypothesis.

Theorem 2.5.2. Subsequences of a convergent sequence converge to the same limit as the original sequence.

Given: $a_n \to a$, and $a_{n_k}$ is an arbitrary subsequence of $a_n$.

To Prove: $a_{n_k} \to a$, which means $\forall \varepsilon > 0, \exists N$ s. t. $\forall k \geq N, |a_{n_k} - a| < \varepsilon$

Proof. Let $\varepsilon > 0$. Let $a_k$ be an arbitrary element of $(a_n)$. For the given $\varepsilon$, we know $\exists N$ s. t. $\forall k \geq N, |a_k - a| < \varepsilon$. In addition, for all $k$, we know $n_k \geq k$ (since it is a subsequence). Therefore, $n_k \geq k \geq N$. This means for each $n_k \geq N$ it is also true that $|a_{n_k} - a| < \varepsilon$. Select $N$ such that $N = N$. Then $\forall k \geq N, |a_{n_k} - a| < \varepsilon$. QED.

Statement of purpose. Like the previous theorem, this is also an important theorem in real analysis, but we use this as an opportunity to illustrate how logical statements can be unpacked. This relates to the mathematical work that PISTs engaged in with Q5.

Corollary (Divergence Description). If a sequence diverges, then it is either not monotone or unbounded.

Statement of purpose. The corollary is the contrapositive of Theorem 2.4.2, the explanation that you provide of the relationship between this and the original theorem demonstrates how logic can be used to further unpack the consequences of a statement.

Corollary (Divergence Criterion). If there exists two subsequences of a sequence that converge to two different limits, then the sequence diverges.

Statement of purpose. The corollary is the contrapositive of Theorem 2.5.2, the explanation that you provide of the relationship between this and the original theorem demonstrates how logic can be used to further unpack the consequences of a statement.

Example. Does $s_n = (1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, ...) \to 1$ converge?

Proof. Since one subsequence converges to 1 (e.g., $(s_1, s_3, s_5, s_7, ...)$), and another converges to 0 (e.g., $(s_2, s_4, s_6, s_8, ...)\to 0$, then this sequence diverges (by the divergence criterion corollary).
LOGIC

Q9. When we see the equation, $ax + b = c$ (variable $x$, parameters $a, b, c$) we are making a statement that is, “$x$ is a solution to the equation $ax + b = c$.”

Consider the following two sets of algebraic steps:

<table>
<thead>
<tr>
<th>Step 1:</th>
<th>Step 2:</th>
<th>Step 3:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x + 1 = 15$</td>
<td>$2x = 14$</td>
<td>$x = 7$</td>
</tr>
<tr>
<td>$I$</td>
<td>$I$</td>
<td>$I$</td>
</tr>
</tbody>
</table>

For each set of algebraic steps:

i) Which of the algebraic steps can be connected as if and only if statements? For those that cannot, explain why not, and instead, write them as if-then statements.

ii) Describe how the logical structure informs how one should interpret the final answer as being the "solution(s)" to the original equation.
LOGIC
Additional HW exercises

AE.1. In class, an Algebra II teacher has stated that “if two functions are inverses of each other, then they are reflections over the line $y = x$.” On one of the practice problems in class, a student looks at a graph, and says: “Well, these are reflections over the line $y = x$, so they are inverse functions.” Explain how would you as the teacher respond to the student and why you would respond like that.

AE.2. Rephrase the following exchange into a definition (for an arithmetic sequence) and an if-then statement (about arithmetic sequences). Then decide whether the if-then statement is true; if it’s not true, modify the statement so that it is true. Show some of the examples you used to help make your decision, and, if appropriate, explain how the statement relates to any of the real analysis claims from class.

Teacher: One type of sequence that can never converge is an arithmetic sequence.
Student: What’s an arithmetic sequence?
Teacher: One where you add the same number over and over to move from one term to the next, like: $a_{n+1} = a_n + k$, like 4, then, 4+3 is 7, then 7+3 is 10, then 10+3 is 13...

Note that arithmetic sequences, in both recursive and explicit forms, are part of the common core high school curriculum—Standard HSF.BF.A.2.