Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey

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Abstract. We describe a case study in which we investigate the effectiveness of a lecture in advanced mathematics. We first video recorded a lecture delivered by an experienced professor who had a reputation for being an outstanding instructor. Using video recall, we then interviewed the professor to determine the ideas that he intended to convey and how he tried to convey these ideas in this lecture. We also interviewed six students to see what they understood from this lecture. The students did not comprehend the ideas that the professor thought were central to his lecture. Based on our analyses, we propose two factors to account for why students failed to understand these ideas.

1. Introduction

From 1961 to 1963, the esteemed physicist Richard Feynman taught a two-year introductory physics course at the California Institute of Technology. As Feynman was regarded as "a great teacher, perhaps the greatest of his era" (Goodstein & Neugebauer, 1995, p. xix), many members of the Cal Tech faculty attended these lectures and some decided to record Feynman's lectures for posterity. The lectures, published as *The Feynman Lectures on Physics* (Feynman, Leighton, & Sands, 2012), are regarded as classics within the physics community and have been widely praised for their clarity and explanatory value (e.g., Davies, 1995). Yet Goodstein and Neugebauer (1995) alleged that if one looks at the actual pedagogical effectiveness of these lectures, the story becomes complicated. While praising the lectures, Goodstein and Neugebauer claimed that the students' enjoyment did not match the faculty member's enthusiasm:

> "Many of the students dreaded the course, and as the course wore on, attendance by the registered students started dropping alarmingly […] When he [Feynman] thought he was explaining things lucidly to freshman and sophomores, it was not really they who were able to benefit most from what

> he was doing. It was his peers—scientists, physicists, and professors—who would be the main beneficiaries of his magnificent achievement" (p. xxii-xxiii).

This paper investigates the seeming paradox described above in the case of advanced mathematics-- namely that an excellent teacher delivering a lecture that his peers rated as magnificent did not benefit the students.

Many mathematics educators and some mathematicians question the general effectiveness of lectures. In summarizing instruction in abstract algebra, an advanced mathematics course at the tertiary level, Leron and Dubinsky (1995) asserted that there is a broad consensus amongst teachers and students that "the teaching of abstract algebra is a disaster and this remains true almost independently of the quality of the lectures". Indeed, "this is especially true for some excellent instructors" whose "lectures are truly masterpieces" (p. 227). Although these comments were specific to abstract algebra, Leron and Dubinsky's arguments could generalize to lectures in any other subject in advanced mathematics. Thurston (1994) was also critical of lectures in advanced mathematics, noting that "mathematicians have developed habits of communication that are often dysfunctional" (p. 165). According to Thurston, these poor communication habits manifest themselves in mathematics lectures, in which,

> "we go through the motions of saying what students 'ought' to learn while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models […] We assume that the problem is with the students rather than with communication: that the students either just don't have what it takes, or else just don't care." (p. 166)

These comments illustrate a widely held belief amongst mathematics educators and some mathematicians: most lectures in advanced mathematics are ineffective for developing students' understanding of mathematics (e.g., Davis & Hersh, 1981; Dreyfus, 1991; Leron, 1983; Rosenthal, 1995; Rowland, 2001). Our goal in this paper is to explain how students attending a mathematics lecture can fail to understand the main points that the professor is trying to convey.

To do so, we present a case study of a professor presenting a proof in a real analysis classroom and analyze both the professor's and his students' perceptions of this proof.

2. Literature review

2. 1. Received views of lectures in advanced mathematics

Although empirical studies on the teaching of advanced mathematics are scarce (Speer, Smith, & Horvarth, 2010), there has been a fair amount written on this topic based on personal experience and shared opinion. In lamenting the quality of instruction in university mathematics courses, Davis and Hersh (1981) contended "a typical lecture in advanced mathematics… consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (p. 151). Examples of mathematical concepts will be "parenthetical and in brief" (p. 151). Similarly, Dreyfus (1990) wrote that the typical mathematics instructor teaches "almost exclusively the one very convenient and important aspect which has been described above, namely the polished formalism, which so often follows the sequence theorem-proof-application" (p. 27).

This instructional paradigm has been criticized on a number of grounds. Perhaps the most common complaint is that the domination of definitions, theorems, and proofs in mathematics lectures leads the lecturer to pay scant attention to other important types of mathematical thinking (e.g., Boero, 2007; Davis & Hersh, 1981; Dennis & Confrey, 1996; Dreyfus, 1991; Thurston, 1994). Consequently issues such as why mathematical concepts are defined the way they are, how concepts could be understood informally (e.g., graphical interpretations of concepts), and how these proofs could have been constructed are (purportedly) largely ignored in lectures in advanced mathematics. A related critique challenges the notion that mathematical proof, at least as it is traditionally presented, is the best means of communicating mathematical explanation and justification to students (e.g., Leron, 1983; Hersh, 1993; Rowland, 2001; Thurston, 1994). The rigor contained in these proofs can prevent students from having an intuitive understanding of why theorems are true (Hersh, 1993) and discourage them from using informal ways of understanding mathematics to produce proofs (e.g., Dreyfus, 1991).

Several authors have suggested reasons for why mathematicians continue to use the definition-theorem-proof instructional paradigm in advanced mathematics, even though such instruction is believed by many to be ineffective. Some researchers posit that mathematics professors are simply not interested in developing the tools to teach effectively, either because they are rewarded for publishing rather than teaching (e.g., Kline, 1977) or because they believe most students simply are not capable of learning the material (e.g., Leron & Dubinsky, 1995). Others believe mathematicians are more interested in establishing logical truth than promoting understanding (e.g., Hersh, 1993). On the other hand, some mathematicians argued that lectures are a viable way to teach mathematics. For example, Wu (1999) claimed that lectures are necessary to teach the entire required course content. Further, the formal theory of some branches of mathematics is viewed by many as a great accomplishment of the mathematical community that provides rigor and clarification to the field, so it is natural that some mathematicians might want to share these hard won gains with students.

2. 2. Interviews with mathematicians about their teaching

Several researchers have conducted interviews with mathematicians in which they were asked to reflect on their teaching practices in advanced mathematics courses (Alcock, 2010; Harel & Sowder, 2009; Weber, 2012; Yopp, 2011). Collectively, these studies produced several interesting findings. Some mathematicians claimed that although they valued non-formal modes of reasoning, their actual lectures overemphasized the formal aspects of mathematics (Alcock, 2010). Although proofs were accepted as a significant part of lectures in advanced mathematics, mathematics professors usually claimed that they did not present proofs to students to convince them why theorems were true, but to illustrate proof techniques and provide other types of insights (Weber, 2012; Yopp, 2011).

Researchers have also presented mathematicians with pedagogical situations to see how they would respond to them or account for them. Nardi (2008) presented mathematicians with students' incorrect written work, where mathematicians conjectured that students desired to "appear" mathematical (e.g., use mathematical notation) rather than to be mathematical. Lai, Weber, and Mejia-Ramos (2012) studied what types of factors mathematicians considered important for pedagogical proofs (e.g. proofs appearing in undergraduate mathematics textbooks), finding mathematicians used strategies such as centering equations in proofs to emphasize their importance and avoiding extraneous information that might confuse students or unnecessarily lengthen a proof. Lai and Weber (2014) also reported that mathematicians claimed to value proofs that incorporated diagrams and highlighted important ideas, but sometimes did not present such proofs in their lectures.

2. 3. Case study observations of professors' teaching in advanced mathematics classes

Most research examining the actual teaching of advanced mathematics courses has used a case study methodology. Fukawa-Connelly (2012) found that when one abstract algebra instructor presented proofs, she would model many of the mathematical behaviors associated with proof writing. She also consistently wrote out the logical details of the proof while only saying orally why some of these details needed to be justified (Fukawa-Connelly, in press). In another study, Fukawa-Connelly and Newton (in press) studied the different ways that another mathematician (again an abstract algebra instructor) used examples to instantiate mathematical concepts, a topic also investigated by Mills (2012). Weber (2004) reported a semester-long case study on how one professor taught real analysis in a traditional manner, regularly interviewing him about his teaching practices. He found the professor's practices were based both on a coherent belief system and a good deal of thought. Also, like the professors studied by Fukawa-Connelly and Newton (in press) and Mills (2012), the professor in Weber's study would sometimes use informal representations of concepts such as examples and diagrams to help students understand the definitions, theorems, and proofs in the course.

2. 4. Gaps in the literature that this study will address

Based on a systematic review of the literature of teaching in university mathematics courses, Speer, Smith, and Horvarth (2010) noted that there was only a single published study (Weber, 2004) in which a researcher both observed a mathematics professor teaching an advanced mathematics course and interviewed the professor about his or her intentions. They claim more research in this area is needed. Similarly, Mejia-Ramos and Inglis (2009) conducted a bibliographic study on the mathematics education research on proof. These authors found that although there was a substantial body of work on how students and mathematicians constructed proofs and checked purported proofs for correctness, there were no empirical studies on how mathematicians chose to present proofs and few studies on how students understood the proofs that they read. Our current research study addresses these voids in the literature.

We observe that there appears to be some discrepancies between the received view of mathematical lectures reported in section 2.1 and the empirical studies described in section 2.2 and 2.3. For instance, some claimed that mathematicians do not use examples of mathematical concepts in their lectures (e.g., Davis & Hersh, 1981), yet Fukawa-Connelly (in press) and Mills (2012), presented case studies of professors doing this. Hersh (1993) worried that mathematics professors used proof only as a tool for conviction but not explanation, but the interview data of Yopp (2011) and Weber (2012) found that mathematicians did not consider conviction to be an important reason for presenting proofs; rather professors claimed to focus on things such as providing explanation and illustrating proof methods. Weber (2004) found that the professor he was studying was not teaching in a definition-theorem-proof paradigm out of habit or apathy, but was basing his teaching methods on a coherent belief system and a good deal of thought. We do not wish to imply that the received views are necessarily incorrect. It is important to note that the empirical data either relies on self-report (in the interview studies) or small samples (the case studies of instruction) or both. Further, mathematicians who agree to discuss their pedagogy or have their teaching observed might not be representative of most mathematics professors. What these inconsistencies do suggest is that as a field, we lack a robust understanding of how lectures in advanced mathematics are taught and why students do not always learn what is intended by the professor.

3. Theoretical perspectives

3. 1. What mathematics can be learned from a proof?

In this paper, we use de Villiers' (1990) purposes of proof to categorize what students could and do learn from the proofs presented in their mathematics lectures. According to de Villiers (1990), mathematicians engage in the activity of proving for five reasons:

Verification: Proofs are written so that each new statement in a proof is either a premise or a necessary logical consequence of previous assertions. In producing or studying a proof, one can verify that the conclusion of a theorem being proven is a necessary logical consequence of the premises of that theorem (c.f., Duval, 2007). Thus, one reason that mathematicians produce and read proofs of theorems is so they can verify that a theorem is true. While this is clearly an important function of proof, de Villiers (1990) emphasized that this was not the primary function of proof in mathematics (c.f., Mejia-Ramos & Weber, 2014; Weber, 2008; Weber & Mejia-Ramos, 2011). Similarly, some mathematics educators question the value of proofs in the classroom that merely convince students that a theorem is true, but provide no other insights (e.g., Hanna, 1990; Hersh, 1993).

Explanation: Good proofs do not only establish *that* a theorem is true, they also explain *why* it is true. Hanna (1990) and Hersh (1993) argued that explanation should be the primary function of proof in the classroom. Weber (2010) suggested that for pedagogical purposes, one could view a proof as explanatory if students are able to relate the statements in the proof to informal representations of mathematical concepts (such as diagrams, graphs, or kinesthetic motions) that are internally meaningful to the audience of the proof. (See Raman, 2003, for a similar analysis and potential epistemological benefits of such proofs).

Discovery: Good proofs often introduce new ideas or methods that can be extrapolated to discover new mathematics and prove other theorems. Indeed, mathematicians claim that the primary reason that they read published proofs is to identify methods that will help them solve problems that they are working on (Mejia-Ramos & Weber, 2014; Weber & Mejia-Ramos, 2011; see also Rav, 1999). Hanna and Barbeau (2008) argued that classroom proofs could be more pedagogically valuable for students if they introduced new problem-solving methods as well.

Communication: The convention of establishing theorems using proofs provides mathematicians with a shared knowledge and agreed upon norms for argumentation that facilitate debate and resolutions to conflict.

Systematization: In some cases, proof can be used not to verify new results, but rather to verify, or show how, new definitions or axiom systems can account for results that are known to be true (c.f., Weber, 2002).

3. 2. How students learn from lectures

To frame our analysis of how students attend to and learn from lectures, we adopt the framework used by Suritsky and Hughes (1991) and Williams and Eggert (2002). According to these authors, learning from a lecture requires a student to have four broad skills, the first three of which are inter-related and contiguous: listening (i.e., paying attention), encoding, recording the points that an instructor makes in written form (i.e., note-taking), and reviewing. In this paper, we will primarily focus on encoding and note-taking.

Encoding the lecture content involves three phases: (i) understanding each lecture point or idea, (ii) integrating a new point with previous points in the lecture, and (iii) integrating a new point with one's prior knowledge (Armbruster, 2000). With respect to (i), as Williams and Eggert (2002) observed that in most lectures, "it is possible to hear what an instructor says, even repeat what an instructor said, with minimal understanding of the instructor's comment" (p. 175). We expect this problem to be more pronounced in mathematics lectures where the conceptual constructs of the lecture may be unfamiliar to students and abstracted away from their direct perception. One critical ability to learn in a mathematics lecture is to recognize the gist of what the professor has said and rephrase the points that the professor makes using one's own words (Kiewra, 1985).

Note taking is also considered a central part of learning from lectures, both because the process of recording notes facilitates comprehension of the points that the professor is asserting and a written account of these points enables the student to review and reconstruct these ideas at a later time (Kiewra, 1985). If a student does not write down a particular point, he or she will usually have difficulty recalling the point, with some researchers estimating that students will only be able to do so as little as five percent of the time (Einstein, Morris, & Smith, 1985). The typical lecturer speaks at a rate of between 100 and 125 words per minute (Wong, 2014) while the typical college student can record only about 20 words per minute (Kiewra, 1987). Consequently, learning from lectures requires the student to *prioritize* what points they choose to record. To do so, the students need to distinguish points that are central to the concepts being discussed from those that are superficial.

The theoretical frame and results above were in the context of domain-general college lectures, not lectures in advanced mathematics *per se*. Nonetheless, this analysis suggests two potential barriers to mathematics majors learning from these lectures. First, students may fail to encode the content of the lecture by being unable to comprehend the main points that the professor is intending to make and to connect these points to their own prior knowledge. Second, even if students are able to understand the main ideas of the lecture, they may be unable to distinguish these important ideas from other superficial comments, which may in turn lead students to focus on minutia rather than on the main points that the lecturer intended to make. In the case study in this paper, we observed multiple instances in which mathematics majors failed to comprehend the lecture for these two reasons. The contribution of this paper is to offer a theoretical rationale for how and why these phenomena occurred.

To avoid misinterpretation, we are not claiming to offer an exhaustive list of reasons why students might not learn from a lecture. Students may not comprehend a mathematical lecture for a large number of reasons, such as not paying attention for part of the lecture, not spending sufficient time reviewing their notes outside of class (c.f., Weber & Mejia-Ramos, 2014), or lacking the opportunity to engage in the difficult process of accommodation and reflective abstraction that mathematical learning often requires (c.f., Leron & Dubinsky, 1995). Rather, the goal in this paper is to give a theoretical account of some of the reasons that students might not understand lectures. We also do not believe this theoretical frame offers a comprehensive account of how learning from lectures occurs. Significantly more theoretical work is needed on the processes involved in how students interpret what their professors are saying, connect it to their own mathematical understanding, and refine their mathematical models as a consequence of what they have heard or interpreted. Such issues are clearly important (and we encourage more research in this area), but are beyond the scope of the methodology of our study. What this theoretical frame offers is a way to conceptualize and investigate a subset of the reasons for why mathematics majors might not understand what their professors try to convey in lecture. As research in this area is currently sparse, we view this work as an important first step toward addressing a much broader issue.

4. Methods

4. 1. Rationale for the study

Firestone (1993) distinguished between two goals of qualitative research: forming sample-to-population generalizations and analytic generalizations[1]. With *sample-to-population generalizations*, the researcher examines attributes of a representative sample of a population and uses the tools of statistical inference to extrapolate these attributes to an entire population. To avoid misinterpretation, the purpose of this case study is *not* to form sample-to-population generalizations. We do not claim that the professor whom we study is representative of all mathematics professors. Indeed, this professor is atypical in that he had a reputation for being an excellent lecturer.

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[1] Firestone (1993) also described *case-to-case generalizations* in which the researcher provides a thick, rich description of an intervention and leaves it to the reader to decide if and how aspects of the intervention can inform their own intervention. This type of generalization is not relevant for the current manuscript.

In this paper, we propose *analytic generalizations*, which we interpret as identifying constructs that are useful in interpreting a phenomenon, illustrating how the interactions of these constructs can account for the phenomenon in question, and posing grounded hypotheses that can form the topic for future research. In our case, we want to know what constructs relate to students' inability to comprehend lectures in advanced mathematics and use these constructs to explain how this lack of comprehension occurs.

Given the current state of research on advanced mathematics lectures, we argue that studies building analytic generalizations are appropriate. The quality of a sample-to-population study is dependent upon the quality of the constructs being employed and the hypotheses being tested. Given the dearth of knowledge on the teaching of advanced mathematics (Speer et al., 2010), we believe it is reasonable to first develop a better qualitative understanding of the lecture before engaging in larger scale studies.

4. 2. The lecture

This research took place at a large state university in the northeast United States in a real analysis course. At this university (and most universities in the United States), real analysis is a junior-level course that is required for mathematics majors. We chose to study a section of the course taught by Dr. A (a pseudonym). Dr. A had over three decades of experience teaching collegiate mathematics and regularly taught real analysis. He also received high teaching evaluations from his students and had a reputation among his peers for being an outstanding instructor. We chose to focus on the lecture of a perceived high quality instructor because we believe that this is the lecture students are most likely to comprehend. If comprehension did not occur in this setting, there were likely to be theoretically interesting and informative reasons for students failing to understand this lecture. Studying a poor instructor is likely to lead to rather mundane accounts for why his or her lecture was ineffective, such as the professor being unprepared, disorganized, insufficiently engaging to hold students' attention, and so on.

The lecture that we investigated was videotaped during the eighth week of the 15-week course. During the lecture, the professor was almost exclusively situated between the students, who were sitting in desks, and the blackboard. The video focused exclusively on the actions of the professor, and recorded all his oral comments, what he wrote on the blackboard, and the gestures that he made. Our analysis focuses on Dr. A's proof of the following claim: If a sequence $\{x_n\}$ has the property that there exists a constant r with $0<r<1$ such that $|x_n - x_{n-1}| < r^n$ for any two consecutive terms in the sequence, then $\{x_n\}$ is convergent. This was one of seven proofs presented in the lecture. We chose to study this proof because we felt it was the most conceptually interesting; the other six proofs were either computation-oriented or illustrated a concept by showing an example satisfied the concept definition. We believe the focus was appropriate for this case study because we wanted to see how students understood aspects of the lecture where conceptually rich ideas were conveyed.

To avoid ambiguity, we refer to the *blackboard proof* as the argument that Dr. A inscribed on the blackboard demonstrating that the theorem was true. We refer to the *lecture proof* as the totality of the 10-minute segment, which includes the blackboard proof, but also Dr. A's oral comments and his gestures-- that is, what we perceive to be the totality of the proof. The focus of this study is on the lecture proof.

4. 2. A mathematically knowledgeable audience's interpretation of the lecture

As a first pass through the data, we identified what important mathematical ideas were being conveyed in the lecture and if we felt these ideas were expressed clearly. A main purpose of this analysis was to rule out the possibility that students failed to understand the main points in

Dr. A's lecture because he had been unclear in his presentation, as judged by a mathematically knowledgeable audience. Rather, our goal is to explain why Dr. A's students did not understand points that were clear to an acculturated mathematical audience.

The four authors of the paper viewed the lecture proof individually; three of the authors had master's degrees in mathematics and experience teaching proof-oriented courses in advanced mathematics. Each member of the research team flagged for instances when he or she felt that Dr. A was trying to convey an idea to his students. For each instance, the researcher noted what this idea was and what actions Dr. A did to communicate it to the students. We then coded each idea based on de Villiers' (1990) purposes of proof. If the emphasis of the idea was on verifying that a given statement was true, we coded this content as an instance of *verification*. If Dr. A gave a conceptual explanation for why a theorem was true, we coded this content as *conceptual explanation*. If Dr. A highlighted ideas within the proof that might be useful for discovering or proving other theorems, we coded these ideas as *method*.[2]

We sought independent confirmation of our analysis by asking a mathematics lecturer who was concurrently teaching a different section of real analysis to view the videotape and describe what he thought were the main ideas of the lecture. The first author of the paper met with this lecturer and audio-recorded their meeting. The first author showed the videotape to the lecturer and asked the lecturer to describe any ideas that he thought were being presented; the first author would stop the video if she felt the lecturer needed time to elaborate. After viewing the tape in its entirety, the first author asked the lecturer to summarize the ideas that were presented in the proof. This interview lasted 20 minutes.

4. 3. Dr. A's aims and interpretation of the lecture proof

[2] We did not code for the *communication* and *systematization* purposes as they were not relevant for this proof; the proof in question was not intended to introduce or highlight the norms of proof or to justify the use of a new definition or a given axiomatic system.

After the initial analysis of the text, the first author met individually with Dr. A for an audio-recorded interview. The interviewer first asked Dr. A why he chose to present the theorem and its proof. Dr. A was then asked what he thought were the main ideas he was trying to convey to the class in his proof. Next, Dr. A was shown the video from his lecture of the proof under investigation here. He was asked to stop the video at any point where he was attempting to convey a mathematical idea that he just described. Whenever Dr. A stopped the tape, the interviewer asked how Dr. A was trying to convey these ideas to the students. If Dr. A had not stopped the tape at a point that we had identified as conveying an idea in our previous analysis, we would have shown Dr. A these excerpts after he was finished viewing the tape in its entirety. However this turned out to be unnecessary; Dr. A stopped and described every point in the lecture where we felt a mathematical idea was being conveyed. After viewing the entire lecture, the interviewer asked Dr. A if there were any ideas that he was trying to convey when presenting a general proof to his students and what things he did to convey these points. This interview lasted 75 minutes.

We analyzed Dr. A's comments about the mathematical ideas he was trying to convey using a semi-open coding scheme. If his comments were consistent with what we observed in our analysis of the lecture, we would fold them into the categories that we had previously formed. If not-- that is, if he introduced a new idea or described the intent of his actions in a different way than we had interpreted them -- we would form a new category. These categories were then coded using de Villiers' (1990) purposes as we described above.

4. 4. The students' interpretation of the lecture proof

After the lecture was given, the first author visited Dr. A's course and invited students to participate in a study on how they understood a mathematical lecture. Students were paid a nominal fee (20 US dollars for a one hour interview) for their participation. Six students volunteered to participate in this study.

Two weeks after the lecture[3], the students were interviewed in pairs, as we anticipated that the opportunity to communicate with one another would elicit more comments from the students. We refer to the first pair of interviewed students as Pair 1 and the individual students as S1 (Student 1) and S2, the second pair of interviewed students as Pair 2 and the students as S3 and S4, and the third pair as Pair 3 and the students as S5 and S6. Dr. A was asked to describe the performance of each of the six students who participated in this study. He described S3 as "below average", Student S1 and Student S5 as "average", Student S2 and Student S4 as "above average", and Student S6 as "an A student". Thus, from Dr. A's perspective, the students that we interviewed were collectively above average and covered a wide range of performance. This suggests that these students would be more likely to comprehend the lecture than the average student, while at the same time providing us with a broader insight into why students might fail to comprehend a lecture.

Each pair of students was asked to bring their lecture notes to the interview, which the interviewer photocopied. All interviews were video-recorded. The interview involved four passes to explore students' understanding of the proof. The data collection and intention of each pass through the data is presented in Table 1.

The purpose of the first pass in the student interviews was to see what students could reconstruct from their experience attending a lecture a couple of weeks after the lecture had passed. The students were invited to review their notes, and asked to describe what they thought

[3] We would have preferred to interview the students earlier, but Dr. A forbade us from recruiting students during this time. He gave a mid-term exam the week after this lecture and did not want the students to be distracted from studying for it. Nonetheless, we are grateful that Dr. A generously provided us with his time and access to his class without compensation from us.

were the main ideas of the lecture proof. In the one occurrence that a student (S2) had not taken notes during lecture, he was provided with a copy of everything Dr. A wrote on the board, which was distributed to the rest of the students in the second pass. We note that in this and all subsequent passes in the student interviews, we set a relatively low bar for what counted as encoding the main points that Dr. A was intending to make: we simply assessed whether or not students could identify these main ideas and rephrase them using their own words. We argue that students failing this simple encoding assessment will most likely fail any assessment of deeper understanding of such ideas.

The purpose of the second pass in the student interviews was to estimate what students understood immediately after viewing the lecture proof in real time. The students were shown a video recording of the proof that they observed. They were asked to behave as if they were in a mathematical lecture, including taking notes. However, they were also given a copy of everything that was written on the board, as this could be difficult to read while watching the video recording. After watching the video, they were then asked to describe what they thought the professor was trying to convey when he was presenting the proof. We note that in this situation, students were placed in a privileged environment that enhanced their chances of understanding the main ideas of Dr.A's presentation. They were watching the lecture for a second time after they had studied the material for a test. The students knew they would be asked about what ideas Dr. A was trying to convey in the lecture immediately after viewing the tape, increasing the likelihood that they would be paying close attention to the lecture. Hence, we argue that any failure of comprehension that occurred in these idealized conditions would be likely occur in the classroom as well.

Table 1.
Summary of four passes through the lecture proof with students

Pass	Data collection	Purpose
Pass 1	Participants recalled what they learned from the proof by reviewing their notes.	We wanted to see what participants could reconstruct from a lecture proof after some time had passed.
Pass 2	Participants viewed the video recorded presentation of the proof, took notes, and were asked what they learned and what the instructor was attempting to convey.	We wanted to see what participants understood immediately after viewing the proof.
Pass 3	Participants watched short specific clips from the proof and were asked what ideas (if any) Dr. A could have intended to convey in those clips.	We were investigating whether participants had the ability to encode what Dr. A had identified as an important idea in his presentation of the proof.
Pass 4	Participants were asked whether particular content highlighted by Dr. A in his interview could be gleaned from the proof they just watched.	We were exploring if, and how, participants understood the main ideas that Dr. A claimed he was trying to convey with this proof.

The purpose of the third pass in student interviews was to determine if the students could encode what Dr. A was saying at each point that Dr. A felt he was conveying an important mathematical idea. The students were shown each of the particular clips that Dr. A flagged as points where he was trying to convey a specific idea. The participants were told these excerpts were places where Dr. A thought he emphasized some important ideas in the proof. The participants were asked to describe what they thought about these clips, but the interviewer added, "it's acceptable to say that you don't see anything. I don't want to encourage guessing". By asking for their reaction to such a short clip, we are reducing the possibility that the students did not describe the ideas presented in the clip because they forgot what they observed or did not prioritize these ideas.

The purpose of the fourth pass through the student interviews was to see if they were aware that Dr. A had tried to convey specific ideas as well as to ask about their interpretations of those ideas. In this pass, the interviewer identified each idea that Dr. A claimed he was trying to convey in this proof. The participants were told that some versions of this proof could convey this specific idea and asked if they thought the proof that they just observed did this. For instance, in his interview, Dr. A stressed that it was useful to think of the terms of the sequence as approximations of the limit and the epsilon in the definition of the limit as representing the error of the approximation. The question the interviewer asked with respect to this metaphor was, "another thing you might have gotten from this proof is the idea that the epsilon used is the error. Is that something that you got from this presentation?" If the participants answered affirmatively, the interviewer encouraged them to describe how the proof conveyed this idea and how they understood it. The goal was to see how participants understood the main points that Dr. A was trying to convey if asked specifically about them.

In analyzing the first and second passes through the data, we used open coding to determine what these three pairs of students identified as the most important ideas in the proof; our initial categories of the content were those formulated by our research team and claimed by Dr. A, but we formed new categories if the students' comments did not fit within our initial framework. In the third pass through the data, we compared students' interpretation of the video clips to the meaning that Dr. A ascribed to them in his interview. In the fourth pass through the data, we analyzed if students' understanding of specific ideas of the proof were aligned with Dr. A's intentions. Hence, this analysis provides insight into the extent to which students understood the ideas that Dr. A felt were important.

5. Results

5. 1. Summary of the lecture

In Dr. A's interview, he was shown a video recording of his lecture and asked to stop the recording at every instance in which he was trying to convey an idea to the students. He stopped the tape seven times and described five specific ideas: (1) Cauchy sequences can be understood geometrically as sequences that are "bunching up", (2) if one does not have a candidate for the limit of a given sequence, one can still show the sequence is convergent by showing it is Cauchy, (3) there is a common structure for writing a proof that shows a sequence is Cauchy, (4) the triangle inequality is useful for proving that summations in an absolute value can be kept small, and (5) the geometric series formula is part of the mathematical toolbox to keep some desired quantities small (appropriate for analysis courses). In our research team's analysis of the lecture, we noted that Dr, A was intending to convey points (2), (3), (4), and (5) and the instructor of another real analysis section claimed Dr. A was trying to convey points (2), (3), and (5).

Table 2.
Summary of the ideas that Dr. A claimed to convey in the lecture proof

Idea	Type of idea	How this idea was conveyed
Cauchy sequences can be understood as sequences that "bunch up"	Conceptual explanation	Lines 3-7. Dr. A states orally, with gestures, how these sequences "bunch up".
One can prove a sequence with an unknown limit is convergent by showing it is Cauchy	Method	Lines 9-17. Lines 25-28. Dr. A states orally why Cauchy's theorem is both useful and necessary to prove this theorem
How one sets up a proof that shows a sequence is Cauchy	Method, Verification	Lines 21-28. Dr. A writes out the structure of the proof, explaining what needs to be shown to prove a sequence is Cauchy
The triangle inequality is useful in proving series in absolute value formulae are small	Method	Lines 39-46. Dr. A states orally that the triangle inequality is used "over and over again" in these proofs
The geometric series formula is part of The mathematical toolbox to keep some desired quantities small	Method	Line 59. Dr. A states orally that the geometric series formula needs to be in the students' mathematical toolbox

We coded point (1) as providing a conceptual explanation for why Cauchy sequences converge, points (2), (3), (4), and (5) as methodological ideas that that might be useful for proving other theorems, and point (3) as also pertaining to the verification that a given statement was true (i.e.

the verification that a given sequence is Cauchy). Neither our research team nor the other instructor of real analysis identified Dr. A as conveying ideas that Dr. A did not identify himself. We present a summary of this lecture content in Table 2.[4] For brevity, we do not report a comprehensive analysis of how we analyzed this data, but a more rigorous and complete treatment of these data is presented in the supplementary materials.

Table 3.
Summary of when student pairs described the ideas that Dr. A was attempting to convey

Idea	Pair 1	Pair 2	Pair 3
Cauchy sequences can be understood as sequences that "bunch up"	Pass 3	Pass 4	Pass 3
One can prove a sequence with an unknown limit is convergent by showing it is Cauchy	Pass 3	Pass 3	Never
How one sets up a proof that shows a sequence is Cauchy	Pass 4	Pass 2	Pass 4
The triangle inequality is useful in proving series in absolute value formulae are small	Pass 2	Pass 3	Pass 3
The geometric series formula is part of the mathematical toolbox to keep some desired quantities small	Never	Never	Never

In Table 3, we report in which pass of the student interviews that each pair of students successfully identified each of the ideas that Dr. A was trying to convey. No pair of students stated any of the ideas that Dr. A noted in Pass 1, in which they were asked to review their notes and recall what ideas Dr. A was trying to convey. Similarly, students highlighted few of the ideas that Dr. A was trying to convey in Pass 2, immediately after re-watching the video recording of the lecture proof. Thus, even though Dr. A had a reputation for being an excellent lecturer, our

[4] Dr. A listed several metaphors that he wished to convey when presenting these types of proof in general, such as comparing a convergent sequence as a successive approximation for its limit term with the epsilon being a bound for the error of this approximation, but these metaphors were absent from the particular proof that we observed.

research team felt that he conveyed points (2), (3), (4) and (5) clearly, the participants in our study were rated (collectively) as being above average, and the participants knew they would be asked about the main ideas in the lecture proof immediately after watching it for a second time, the participants for the most part were still unable to state the main points that Dr. A was trying to convey. If students were not at least recognizing the main points that Dr. A was trying to convey in these favorable settings, it seems likely that there would be wider gaps in comprehension in more typical and realistic settings. In the remainder of this section, we will explain some of the reasons that comprehension failed to occur.

5. 2. Distinguishing the blackboard proof from the rest of the lecture proof

5. 2. 1. When to use the fact that Cauchy sequences converge. Here we discuss why the participants did not highlight point (2), that if one does not have a candidate for the limit of a given sequence, one can still show the sequence is convergent by showing it is Cauchy, in the first two passes of our interviews with them. At three points during the lecture proof, Dr. A made this point. In lines 9 to 17, Dr. A asked the students what types of sequences converge even if the limit cannot be determined. In these excerpts, he said,

> Dr. A: There's no mention of what the definition is of the sequence, so there's no way we're going to be able to verify the definition limit of a convergent sequence, where we have to produce the limit. So what do we do? […] What kind of sequences do we know converge even if we don't know what their limits are? It begins with a 'c'.
> Student: Cauchy.
> Dr. A: Cauchy! We'll show it's a Cauchy sequence.

In lines 17 through 19, Dr. A reiterated this point:

> Dr. A: We will show that this sequence converges by showing that it is a Cauchy sequence [writes this sentence on the board as he says it aloud, then turns around to face class]. A Cauchy sequence is defined without any mention of limit.

In lines 25 to 28, after writing the Cauchy sequence definition, Dr. A again made this point:

> Dr. A: This is how we prove it is a Cauchy sequence. See there is no mention of how the terms of the sequence are defined. There is no way in which we would be able to propose a limit *L*. So we have no way of proceeding except for showing that it is a Cauchy sequence or a contractive sequence.

From our perspective, this was the most important idea in the proof. It was stated repeatedly and clearly. The other instructor viewing the proof agreed, calling it "the main objective" of presenting the proof. No pairs of students mentioned this through the first two passes of the data. However, in Pass 3, both Pair 1 and Pair 2 identified this point when shown the specific clips where Dr. A discussed it. For instance, when Pair 1 was shown the first excerpt in this section, S1 said, "we should recognize it, like to figure out it's a Cauchy, we should know that it's converging, but it's limit is not necessarily given. So that we recognize it instantly" and S2 said, "Because we don't have the limit here, or we have no way of figuring out what the limit is. All we have is them in relation to each other. Cauchy makes sense". Thus, both Pair 1 and Pair 2 demonstrated that they were capable of encoding what Dr. A was saying when shown these segments in isolation, but did not identify this key piece of information after watching the proof in its entirety.

5. 2. 2. Our interpretation to account for this phenomenon. We noted that for Dr. A's points (1), (2), (4), and (5), he stated these points aloud but he did not write them on the blackboard. The only thing written on the blackboard was a polished proof of the proposition[5]. Hence, the main ideas that Dr. A was trying to convey were usually only stated orally. When describing his lecturing approach in his interview, Dr. A remarked:

> Dr. A: By asking questions, and asking people by names, they will have their minds alert, saying 'he might ask me, I'd better think about what's going on.' Now we all fall asleep in classes at times, so it's not clear you're always going to be alert. But hopefully, that *if the lecture is going to be of use to people, that during the lecture at times their minds are picking up something useful. Otherwise they're just copying off the board*, which is what we always do sometimes too. But, it makes it a little

[5] We conjecture the only reason that (3) was written on the blackboard was because it was part of the proof.

> more exciting for me to be able to ask questions and talk to the class rather than stand up there and write stuff on the board. (italics are our emphasis).

There are several important ideas here. First, in the italicized section, Dr. A stressed that students cannot learn by "just copying off the board". Dr. A talks to the class, implying that what he says orally matters. Second, because he is aware that students may not be actively paying attention (as "we all fall asleep in class sometimes"), he will ask students questions to make them alert. Indeed, in this lecture, he asked questions immediately before conveying points (2), (3), (4), and (5), which were presumably intended by Dr. A to grab the students' attention.

When we looked at the students' notes, we found that only one student, S1, recorded any of Dr. A's oral comments in her notes. S3, S4, S5, and S6's notes consisted of near verbatim transcriptions of what was written on the blackboard, while S2 did not take notes at all. This is consistent with the general literature on students' note-taking during lectures, showing that students were far more likely to record what was written on the blackboard than what was stated only orally (e.g., Johnston & Su, 1994). As noted earlier, students rarely recall points from a lecture that they do not write down in their notes (e.g., Einstein et al, 1985; Kiewra, 2002), which can explain why Pair 1 and Pair 2 did not recall Dr. A's point (2) after watching the proof, even though Dr. A stated this point three times and the participants could state the gist of Dr. A's comments if shown them in isolation.

5. 2. 3. Discussion. When giving a lecture, mathematics professors are faced with a myriad of goals, making their job challenging. Two particular goals include (i) helping their students realize what an acceptable proof is (i.e., understanding what constitutes an acceptable proof *product*) and (ii) teaching their students about the processes of writing a proof (i.e., understanding the *process* to produce this product). These goals can be in conflict with one another. Students often think a proof should be a description of their problem-solving process while a proof itself focuses on validation, not describing the decision-making process that was used to produce it (Selden & Selden, 2013). To manage this tension, we believe Dr. A elected to have the blackboard proof represent the proof product while describing the process of creating the proof aloud. Thus, both product and process were being conveyed to his class, but the two types of content were not conflated.

The students' job in a mathematics lecture is also challenging. They are learning about new and complex ideas. As the professor is speaking at a faster pace than the students can write, the students must prioritize certain ideas over others. It is natural for the students to focus on what is written on the blackboard; this is a traditional way by which teachers emphasize importance and written comments have a permanence that oral comments lack. The consequence of students behaving in this way during Dr. A's lectures is that they were not recording what Dr. A considered to be the main points of his lecture. As a result, these ideas were not recorded and may have been ignored.

5. 3. Mathematical idioms

5. 3. 1. Toolbox of techniques to keep things small. Dr. A highlighted two points in the lecture where he was illustrating that real analysis proofs involved using the fact that one quantity was small to show that another quantity was small. In lines 33 through 37, he said:

> Dr. A: Now once again we ask the question. If we were to show this is small, we must represent it in terms of what we know is small. Well what do you know is small? For n large enough [gestures toward the statement of the theorem], the difference between two consecutive terms is small. [Turns and faces the blackboard]. So what we must do is represent that as a sum of consecutive terms.

Later, in lines 53 through 61, the following exchange took place.

> Dr. A: So let's factor out the smallest term, r^n. What's left is $1 + r + r^2 +$ up to r^{m-n}. [Writes this equation on the blackboard as he speaks]. Now we know this is small [circles r^n] now what can we say about this expression right here? [points to and circles the geometric series $1 + r + r^2 + \ldots + r^{m-n}$,

> then turns around and faces the class]. Anybody have a vague idea? I'll give you a hint: Calculus II. Thirty or forty years ago.
>
> Student: Geometric series.
>
> Dr. A: Geometric series! [Turns and faces the blackboard]. You have to always keep a geometric series in your toolbox. So it's going to be less than r^n, this [gestures towards the geometric series written on the blackboard] then is less than sum from k=0 to infinity of r^k. And now we need to know the formula of a sum of a geometric series.

Both our research team and the other course instructor thought an important theme of the proof was that in real analysis proofs, one wanted to show that one quantity was small when being given that another quantity was small. Dr. A corroborated that this was his intention during his interview. When asked what he meant by mathematical toolbox in this context, Dr. A replied:

> Dr. A: Once you get into the area where you're doing approximations, you can't do equal, equal, equal. You have to have bounds, bounds, bounds […] *The objective is to show how bounds, using the triangle inequality, can be used to show that something is small using information that they're given is small.* And this instance turns out that the information which is small is given in a form that allows us to use the geometric series as a bound. (italics are our emphasis).

The participants did not cite the notion of making terms small in Pass 1 or Pass 2 through the interview. In Pass 3, when participants were shown the first clip in this subsection (lines 33 through 37), they focused on the algebraic manipulations taking place. For instance, in Pair 2, S4 said, "basically manipulating the information that we're given so that we can show that a sequence fits the definition". In Pair 3, S6 said: "Given on the problem to see like what we could, how we can manipulate the problem statement. Just how we can start the proof in general". None of the six participants used the word "small" or any synonym for small in their response. When participants were shown the second clip (lines 53 through 61), all three pairs of students remarked that Dr. A was trying to convey that one can use what they learned from calculus courses in real analysis proofs. Dr. A did remark on this in the clip ("I'll give you a hint. Calculus II") but he did not cite this in our interview as an idea he was trying to convey.

In Pass 4, each pair of participants was asked, “one last thing you might get from this proof is that mathematics students need to have a toolbox of ideas that help them to prove things are small. Is this something that you got from this presentation?” All six participants answered yes. However, when asked to elaborate, none mentioned inequalities or making quantities small. In their responses, the participants tended to focus on the word “toolbox” and describe general techniques they learned from the proof. For instance, S2 described Cauchy sequences becoming part of his toolbox and S3 discussed the format for beginning the proof as being in his “toolbox memory area”. Only one student, S5, mentioned the word “small” in his response. In the following excerpt, we can see that S5 was not using small in terms of a magnitude of a quantity, as Dr. A intended.

S5: We can use Mathematica, or like a tool to convert to make something small.
I: So right so mathematics students need to have a toolbox of ideas to help them prove things are small.
S5: Things are small. Oh you mean that they're not so complicated. When you say that things are small?
I: No I mean like in terms of convergent sequences. Is that something that you think you got from this presentation?
S5: I mean, in terms of simplifying them and deriving for approximating the answer, I think it's on the path, it's like it's working.

By listing Mathematica (a computer algebra system commonly used in college calculus classes but not real analysis), S5 is referring to general mathematical tools for performing calculations and solving problems. This toolbox did not appear to include techniques for working with inequalities or keeping quantities small, which was part of Dr. A’s intention. S5’s response to the interviewer’s follow-up question revealed that he did not know what Dr. A meant by “small” in this context, guessing that it means a not complicated, or simplified, equation, rather than a quantity with a small magnitude.

5.3.2. Our interpretation of this phenomenon. The participants were not able to describe the gist of what Dr. A intended to communicate in the two clips above. Even though Dr. A's intent was clear to our research team and another instructor of the course, it was not transparent to the students. We postulate that the following two factors accounted for the participants' lack of comprehension. The first was that the participants did not share Dr. A's understanding for what Dr. A meant by the words "small" and "toolbox". We refer to the use of *small* in this context as a mathematical idiom that expresses a technical mathematical idea using informal English language. In some mathematical contexts, "small" might mean a negative number with a large magnitude, a number with an inconsequential magnitude compared to other magnitudes in an expression (if one is dealing with millions of dollars, a dollar value less than ten would be trivially small), or a number with a tiny magnitude in an absolute sense (one billionth). But that is not what small means in this context. Here small means *arbitrarily small*. Oehrtman (2009) demonstrated that in an introductory calculus course, students viewed notions of arbitrarily small and sufficiently small as meaning "very small" and it did not play a role in how they metaphorically understood limits. Even though Dr. A uttered the word small eight times in his lecture, this did not factor into how the participants interpreted the lecture. A similar case can be made for the word "toolbox". By toolbox, Dr. A may have meant a broad set of problem-solving and proving techniques but he clearly wished to include techniques for working with inequalities and techniques for transforming an expression so that the resulting expression, in a sense, stays small. The participants interpreted "toolbox" as meaning any technique useful for doing mathematics, which is a reasonable interpretation, just not the idea Dr. A intended to convey with this proof.

We posit that a second factor inhibiting comprehension was participants' orientation toward viewing proofs as a series of calculations rather than the application of a holistic method to solve a given problem. Students' propensity to focus on calculation rather than the overarching ideas of the proof has been documented in the proof reading literature (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003). In particular, in a large scale survey, we found that the majority of mathematics majors believe understanding a proof consists entirely of being able to say how each new statement in a proof follows logically from previous statements (Weber & Mejia-Ramos, 2014), even though most mathematicians believe that a key insight that proofs can provide is a more general method for proving a class of theorems (Mejia-Ramos & Weber, 2014; Weber & Mejia-Ramos, 2011) and try to convey this to their students when they teach (Weber, 2012). Although Dr. A prefaced the first clip in the subsection with the question, "If we were to show this is small, we must represent it in terms of what we know is small. Well what do you know is small?", the participants thought the clip was about performing algebraic manipulations. S5 illustrated a rather extreme focus on calculation when he thought making quantities small involved making the lengths of expressions small.

5.3.3. Discussion. Dr. A engaged in ostensibly useful practices to promote conceptual understanding in his lecture. He phrased technical ideas in intuitive terms using mathematical idioms such as "small" and he used the notion of smallness to describe his overarching method for approaching this proof. He prefaced each clip with a question, a practice that he said he used to increase the chance that they will be paying attention when he stated these points. The intent of these clips was clear to both our research team and another instructor who viewed the lecture. Nonetheless, the participants did not grasp the gist of what Dr. A was asserting when he discussed the method of the proof, in part because they did not understand what he meant by small and they did not view his discussion of smallness as being important to the message he was conveying, a phenomenon also observed by Oehrtman (2009). Further, when participants watched the clip of lines 33 through 37, they did not describe the overarching proof method that Dr. A described but the algebraic manipulations in which he engaged. We interpreted the algebraic manipulations as being in support of the plan he laid out, while the participants interpreted these algebraic manipulations as the main point of the clip. While informal explanations of proving methods are considered useful by those who know the subject, this sub-section illustrates that these might not be useful to students if they do not have a shared understanding of the mathematical idioms employed and they do not view such explanations as important. Although the mathematical idioms are expressed informally, their meaning is often precise and complex. It is thus not surprising that students might not grasp this meaning.

6. Discussion

6. 1. Summary of main results

In this paper, we have described students' reactions to a lecture proof that we believed would be considered of high quality by most traditional measures. In the lecture, Dr. A attempted to convey the motivation behind the proof that he was writing and be explicit about an important heuristic that students could take from this proof. He used an interesting pedagogical strategy of posing a question to students to engage them in the lecture prior to making an important point. Several of the important points were made multiple times. Both our research team and another course instructor recognized what Dr. A was attempting to convey and thought that he conveyed these ideas clearly. This was consistent with Dr. A's reputation as an effective instructor and his record of receiving high student evaluations. To avoid misinterpretation, our purpose in this paper was not to critique Dr. A, but rather to show how even an excellent instructor might not be understood by his students.

For the most part, the participants that we interviewed failed to grasp many of the points that Dr. A emphasized as the most important parts of the lecture, even after viewing the lecture for a second time. It is not the case that students did not learn *anything* from this proof. After watching the proof, all pairs of students discussed how the proof illustrated new ways to simplify algebraic expressions and demonstrated that they could use prior knowledge from calculus proofs in a real analysis setting. These are useful points. We also think it would be unreasonable to expect every student to fully understand every aspect of a lecture proof. Students may come to acquire these ideas incrementally and can benefit from seeing the same point across multiple proofs, an idea expressed by Dr. A in his interview and in our other interviews with mathematicians (Weber, 2004, 2012). Nonetheless, we still find it disappointing that the participants in our study failed to grasp nearly all of the points that Dr. A himself identified as the key ideas of his lecture. Further, these participants were unlikely to reconstruct these points at a later time as most did not record in their notes the ideas that Dr. A thought were important.

We acknowledge that the methodology used in this study did not represent a typical students' experience in an ordinary lecture. We had above average students watching a good lecturer deliver a conceptually rich proof for a second time in a setting where they were given the blackboard proof prior to watching the lecture. Still, we argue that for the most part, the discrepancies between our study and a typical student's experience in an ordinary lecture would, if anything, make it *more likely* that they would develop a more complete comprehension of the lecture. Consequently, this makes students' failure to simply identify and rephrase the main points of lecture more concerning.

This study represents an important illustration that even a lecture that would meet traditional standards of quality might not be understood by students. We identified several factors for why comprehension did not occur in these cases. The most important ideas of Dr. A's lecture were delivered orally but the participants' notes focused on the formal proof written on the blackboard. Dr. A used mathematical idioms such as "small" to describe the motivation behind the proof but participants' did not share Dr. A's understanding of these idioms and did not attend to these conceptual explanations. They instead focused on the algebraic manipulations within the proof.

6. 2. Limitations of the study

There are two limitations due to the intent and the scope of the study. First, to avoid misinterpretation, we reiterate that we are not claiming sample-to-population generalizations. We do not have a warrant to claim that other lecturers would teach as Dr. A did, nor do we have a warrant to say the proof we studied was typical of one that Dr. A would deliver. (In particular, Dr. A indicated that this particular lecture was unusual in that he usually used diagrams extensively). More research with larger samples is needed to determine the generality of these findings. What this study can provide is constructs to consider and grounded hypotheses to test.

The second issue concerns the scope of the issues that we explored. We explored students' reactions to a lecture proof, both after viewing it in its entirety and watching individual clips of it. This only addresses a fraction of the issues on how students understand lectures in advanced mathematics. Issues about how often they pay attention, and if and how they review their notes outside of class are acknowledged as important (although these are also largely ignored in other disciplines, Williams & Eggert, 2002). We also did not cover other important aspects of lecture outside of proof presentation, including presenting definitions as well as the use of examples and diagrams. Some of these topics have been addressed in other studies (e.g., Fukawa-Connelly & Newton, in press; Mills, 2012; Weber, 2014), but students' perceptions of these aspects of lectures remains largely unexplored. More theoretically driven work on how meaning is constructed in advanced mathematics and how this can or cannot occur in lectures is urgently needed. The current study is important because lectures are the dominant means by which advanced mathematics courses are taught yet there is limited research on this crucial topic in mathematics education (Speer et al, 2010), but it is only a first step in a larger enterprise.

6. 3. Directions for future research

We suggest two broad areas in future research. The first is to investigate further the issues raised in the paper. In particular, how common is Dr. A's practice of only writing the formal proof on the blackboard and stating methodological and conceptual ideas only orally? How common is students' practice of not recording comments that are only stated orally? If these two patterns of behavior are common, this could offer an important factor limiting comprehension for lectures. Studies with larger samples would be needed to address these questions. Regarding mathematical idioms, what are the common idioms used in lectures in advanced mathematics? To what extent is idiom usage consistent and shared across lectures and to what extent is idiom usage idiosyncratic? How are these idioms commonly understood by students? How are these idioms introduced in the lecture and how does this affect student understanding?

The second broad area of research is exploring areas of lecture that were not addressed in this study. The theoretical frame for our study focused our analysis on identifying areas where comprehension did *not* occur. However, we would need a more nuanced lens to document how comprehension could occur or to address the question of whether comprehension of advanced mathematics is reasonable to expect in a lecture format. In particular, issues such as how participants construct the main idea of what the professor says, how their mental frames interact with the professor's utterances to form comprehension, and how comments from the professor could lead students to refine and reorganize their mathematical frames, as well as developing methodologies to address these issues, seem necessary precursors to presenting and testing a model of how students can learn from lectures and we recommend further research in this direction.

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Appendix: Proof transcript

	Time	Verbal	Written
1	0:00:57.0	*(Faces board).* Let's look at the first example.	Example 1
2	0:01:03.0	We're going to look at the following sequence.	
3	0:01:10.0	$\{x_n\}$ for which the difference between the nth and nth minus first term will be less than r^n, where *r* is somewhere between 0 and 1.	Consider $\{x_n\}$ for which $\|x_n - x_{n-1}\| < r^n$, $0 < r < 1$.
4	0:01:26.0	*(Faces class).* So since, when n *(pointing to the r^n written on the board)* gets large this *(pointing to the difference $\|x_n - x_{n-1}\|$ written on the board)* gets very small.	
5	0:01:30.0	It will be that these two *(pointing to the x_n and x_{n-1} terms)* consecutive terms will get closer and closer together as *n* gets larger.	
6	0:01:36.0	Now what you expect as the terms get closer and closer *(brings his two hands together as if to clap them)*, the sequence will converge.	
7	0:01:42.0	What will the limit be?	
8	0:01:44.0	*(Shrugging with hands up)*	
9	0:01:45.0	*(Pointing to the board briefly, indicating what has been written in line 3)* There's no mention of what the definition is of the sequence.	
10	0:01:52.0	So there's no way we're going to be able to verify the definition limit of a convergent sequence, where we have to produce the limit.	
11	0:01:59.0	So what do we do?	
12	0:02:02.0	How can we proceed to show that this is a convergent sequence? Anybody have a guess? *(Pointing to a student who's beginning to speak.)*	
13	0:02:08.0	Student: We could know what's in front of it?	
14	0:02:12.0	Well that's not quite the right term.	

15	0:02:14.0	What kind of sequences do we know converge even if we don't know what their limits are? It begins with "c".	
16	0:02:22.0	Students: Cauchy	
17	0:02:22.0	Cauchy! We will show it's a Cauchy sequence. *(Faces board).*	
18	0:02:29.0	We will show that this sequence converges by showing that it is a Cauchy sequence.	We will show that this sequence converges by showing that it is a Cauchy sequence.
19	0:02:57.0	*(Faces class).* A Cauchy sequence is defined without any mention of limit.	Proof:
20	0:03:02.0	*(Faces board).* Alright so let's look at the proof of this.	
21	0:03:05.0	*(Faces class).* How do we start our proofs about convergence, or Cauchy sequence?	
22	0:03:10.0	*(Faces board).* Let epsilon be greater than zero be given.	Let $\varepsilon > 0$ be given.
23	0:03:16.0	And now we'll state what it is we have to show.	
24	0:03:19.0	We will show that there is an $N(\varepsilon)$ for which $x_m - x_n$would be less than epsilon when *m* and *n* are greater than this number $N(\varepsilon)$.	We will show that there is an $N(\varepsilon)$ for which $\|x_m - x_n\| <$ ε where m and n > $N(\varepsilon)$.
25	0:03:50.0	This is how we prove it is a Cauchy sequence.	
26	0:03:53.0	*(Faces class).* See there is no mention of how the terms of the sequence are defined.	
27	0:03:58.0	There is no way in which we would be able to propose a limit *L*.	
28	0:04:02.0	So we have no way of proceeding except for showing that it is a Cauchy sequence or a contractive sequence.	
29	0:04:10.0	So let's look and see how we proceed. *(Faces board).*	
30	0:04:13.0	Alright. Now lets then set up.	
31	0:04:13.0	Let's consider for *m*, no let's see.	Consider for m
32	0:04:22.0	I guess, *n* greater than *m*. *(Erases what he has just written), m* greater than *n*. $x_m - x_n$.	Consider for $m > n$, $\|x_m - x_n\|$.
33	0:04:31.0	*(Faces class).* Now once again we ask the question.	

34	0:04:33.0	If we were to show this *(pointing to* $\lvert x_m - x_n \rvert$*)* is small, we must represent it in terms of what we know is small.	
35	0:04:40.0	Well what do you know is small?	
36	0:04:43.0	For n large enough *(pointing to* $\lvert x_n - x_{n-1} \rvert < r^n$*, in problem statement on the board)*, the difference between two consecutive terms is small.	
37	0:04:48.0	*(Faces board).* So what we must do is represent that *(pointing to* $\lvert x_m - x_n \rvert$ *just written in line 32)* as a sum of consecutive terms.	
38	0:04:54.0	So, $x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2}$ all the way down to $x_{n+1} - x_n$.	$= \lvert x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} \ldots x_{n+1} - x_n \rvert$
39	0:05:17.0	*(Faces class).* Now what would you do next?	
40	0:05:25.0	Student: The triangle inequality.	
41	0:05:26.0	The triangle inequality; over and over we have to use the triangle inequality.	
42	0:05:29.0	I should point out that in the homework I just passed out.	
43	0:05:31.0	Again a number of you still are not comfortable using the triangle inequality.	
44	0:05:37.0	You have inside an absolute value sign terms with minus signs.	
45	0:05:42.0	And you drop the absolute value signs and still have terms with minus signs.	
46	0:05:47.0	And somehow, you have to be able to get rid of those using the absolute value given by the triangle inequality.	
47	0:05:57.0	*(Faces board).* Alright so use the triangle inequality.	
48	0:05:59.0	So we're going to have less than or equal to $x_m - x_{m-1} + x_{m-1} - x_{m-2}$all the way down to $x_{n+1} - x_n$.	$\leq \lvert x_m - x_{m-1} \rvert + \lvert x_{m-1} - x_{m-2} \rvert + \cdots \lvert x_{n+1} - x_n \rvert$

49	0:06:17.0	And now what we do, is we use the assumed property *(underlines* $\lvert x_n - x_{n-1} \rvert < r^n$ *in problem statement on the board)*, that each of these terms will be less than or equal to -- So each of these terms will be less than or equal to n to the *(points to the n in* x_n *term of* $\lvert x_n - x_{n-1} \rvert < r^n$ *in the problem statement)*, *m*.	$\leq r^m$
50	0:06:35.0	This *(pointing to the |x_m-1 - x_m-2| term)* is going to be less than r^{m-1} and all the way down to r^n.	$+r^{m-1} + \cdots + r^n$
51	0:06:45.0	*(Faces class).* So what we have is a partial series, sum of terms. *(underlines the sum* $r^m + r^{m-1} + \cdots + r^n$*)*	
52	0:06:51.0	*(Faces board).* So let's write that down: $r^n + r^{n+1}$... Nothing's really changing here, but let's write it this way... + r^m	$\leq r^n + r^{n+1} + \cdots + r^{m-1} + r^m$
53	0:07:03.0	So let's factor out the smallest term, r^n. What's left is 1 + r + r^2 + up to r^{m-n}.	$\leq r^n(1 + r + r^2 + \cdots + r^{m-n})$
54	0:07:25.0	Now we know this is small *(circling the* r^n*)*, now what can we say about this expression right here? *(underlining the geometric series* $1 + r + r^2 + \cdots + r^{m-n}$*)*	
55	0:07:22.0	*(Faces class).* Anybody have a vague idea? I'll give you a hint: Calculus II.	
56	0:07:29.0	Student: Geometric series?	
57	0:07:32.0	Thirty or forty years ago. *(Points to the student who spoke.)*	
58	0:07:32.0	Student: Geometric series.	
59	0:07:34.0	Geometric series! *(Faces board).* You have to always keep a geometric series in your toolbox.	
60	0:07:39.0	So it's going to be less than r^n, this *(points quickly to the geometric series* $1 + r + r^2 + \cdots + r^{m-n}$ *written in line 53)* then is less than sum from k=0 to infinity of r^k.	$\leq r^n(\sum_{k=0}^{\infty} r^k)$
61	0:07:47.0	And now we need to know the formula of a sum of a geometric series. *(Faces*	$= r^n(\quad)$ *Leaves parenthesis blank for*

		class).	*now.*
62	0:07:55.0	*(To a student off camera).* You forgot?	
63	0:07:57.0	Student: I forgot	
64	0:07:58.0	It's been ten, fifteen years?	
65	0:07:59.0	Student: Five years	
66	0:08:01.0	Five years. *(Now to another student),* Mr. (Student)?	
67	0:08:02.0	Student: *(mumbles)*	
68	0:08:06.0	Is there anyone who won't use that excuse?	
69	0:08:09.0	Student: One over, I remember something, one over one minus *r*...	
70	0:08:14.0	Oh! Don't say anything more.	
71	0:08:17.0	*(Faces board).* One over one minus *r*.	*Fills in the previous parentheses to get:* $= r^n \left(\frac{1}{1-r}\right)$
72	0:08:20.0	*(Faces class).* Right. Okay?	
73	0:08:22.0	So now, let's see what we have. *(Faces board)*	
74	0:08:34.0	So we have that for $m > n$, this is less than or equal to r^n times $\left(\frac{1}{1-r}\right)$. (*Faces class).*	So we have $\lvert x_m - x_n \rvert \leq r^n \left(\frac{1}{1-r}\right)$
75	0:08:48.0	Now r is some fixed number, it's just some constant. *(pointing to* $r^n \left(\frac{1}{1-r}\right)$*).* *(Faces board).*	
76	0:08:53.0	So we know that since the limit, no *(erases limit),* since *r* is between 0 and 1, what is true of the limit of r^n as n goes to infinity? *(Faces class)*	We know that since $0 < r < 1$ $\lim_{n\to\infty} r^n =$
77	0:09:11.0	Can anybody think of a big number that goes there?	
78	0:09:14.0	A small number, any number! *(pointing to the limit written on the board in line 76).*	
79	0:09:20.0	What happens if you take a number between 0 and 1 and you raise it's power. A half, fourth, eighth...	
80	0:09:25.0	*(Faces board).* It goes to zero!	*Fills in previous line.* 0.

81	0:09:27.0	Alright, so. There exists, using the property of convergence an $N(\varepsilon)$for which r^n, because this is greater than zero, is going to be less than one minus r times epsilon for $n > N(\varepsilon)$	So, there exists an $N(\varepsilon)$for which $0 < r^n < (1-r)\varepsilon$ for $n > N(\varepsilon)$.
82	0:09:57.0	Then for m which is greater than n which is greater than $N(\varepsilon)$, we have, $x_m - x_n$ is going to be *(Pointing quickly to the limit written in line 76)* less than.	Then for $m > n > N(\varepsilon)$ $\|x_m - x_n\| <$
83	0:10:17.0	This is going to be less than *(points to the r^n from the inequality $\|x_m - x_n\| \leq r^n \left(\frac{1}{1-r}\right)$ that was written in line 74)* one minus r epsilon times 1 over 1 minus r, which is epsilon.	$(1-r)\varepsilon \left(\frac{1}{1-r}\right) = \varepsilon$.
84	0:10:27.0	*(Faces class).* And therefore, we have verified that this sequence is a Cauchy sequence. *(Faces board).*	So $\{x_n\}$ is a Cauchy sequence.
85	0:10:32.0	*(Still facing the board and erasing).* Now you can expect to have a problem on the exam where you must verify that a given sequence is Cauchy.	
86	0:10:48.0	That's one of the important results of this part of the course.	