Structure Groups on Pseudo-Riemannian Algebraic Curvature Tensors

CALIFORNIA STATE UNIVERSITY SAN BERNARDINO

Joseph Palmer

California State University San Bernardino, REU Program 2010

Background

The Riemannian curvature tensor is a very important mathematical object in differential geometry and the study of smooth manifolds. This object is actually a tensor field, so restricting it to a single point produces a tensor known as an *Algebraic Curvature Tensor* (ACT). We study possible ACTs in order to gain insight about the overall tensor field. If V is a real vector space of finite dimension n then $R \in \otimes^4 V^*$ is an Algebraic Curvature Tensor if:

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y)$$

$$\text{and } R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0$$

for all $x, y, z, w \in V$. The following class of ACTs is particularly important. Take $x, y, z, w \in V$ and some symmetric bilinear form $\phi \in S^2(V^*)$ and define

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

These are known as the *canonical curvature tensors*. Defining $\mathcal{A}(V)$ as the set of all possible ACTs on V, it is known [3] that

$$\mathcal{A}(V) = span\{R_\phi | \phi \in S^2(V^*)\}.$$

A similar definition and result exist for anti-symmetric forms, but this is not explicitly relevant to our results.

The ordered pair (V, R) of a vector space and an ACT is known as a *weak model space*, often denoted $\mathcal{W}$, and by adding on an inner product $\langle\cdot,\cdot\rangle$ we have the triple $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, which is known as a model space. If all of the objects in the model space can decompose into a direct sum on the same vector spaces, we say that the model space *decomposes*, and write

$$\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R) = (V_1, \langle\cdot,\cdot\rangle_1, R_1) \oplus (V_2, \langle\cdot,\cdot\rangle_2, R_2).$$

Goals and Motivation

In particular, we will be studying the *structure groups* of these objects. The structure group of some covariant tensor, or group of covariant tensors, is defined as the group of endomorphisms on the vector space V which preserve the tensors under precomposition. For example, the structure group of a weak model space $\mathcal{W}$, using the notation $A^*R(x, y, z, w) \equiv R(Ax, Ay, Az, Aw)$, is

$$G_{\mathcal{R}} = \{A \in Gl(n) | A^*R = R\}.$$

We consider the structure groups of several broad classes of model spaces, and explore the connections between the structure of $G_\mathcal{M}$ and the structure of $\mathcal{M}$. In particular, we use $R = R_\phi$ and $R = R_\phi \pm R_\psi$ for $\phi, \psi \in S^2(V^*)$ and we also study structure groups of $\phi \in S^2(V^*)$.

It is often desired to know if a given manifold is locally homogeneous. It is also of interest to determine when manifolds are k-curvature homogeneous, which means that the first k covariant derivatives of R are each locally constant. There exist what are known as *Weyl scalar invariants* built by contractions of R, and in the Riemannian these scalar functions are all constant if and only if the manifold is locally homogenous [4]. This result can not be generalized to the pseudo-Riemannian case. In fact, there exist *vanishing scalar invariant* (VSI) manifolds for which all scalar contractions of R are zero, but not all of these manifolds are locally homogeneous. However, there are classes of pseudo-Riemannian manifolds for which alternate scalar invariants have been found (for an example, see [1]). Thus, by studying the structure group of a given model space, we attempt to form new invariants which will also work in the pseudo-Riemannian case.

Structure Groups of R_ϕ

First consider the simplest case; for some $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ let $R = R_{\langle\cdot,\cdot\rangle}$. It is known that this R corresponds exactly to the case of constant sectional curvature [2]. We give our own proof to the following known result.

Lemma 1. *For the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ (with the inner product positive definite) we have:*

$$G_\mathcal{M} = G_{\langle\cdot,\cdot\rangle} \Leftrightarrow R = cR_{\langle\cdot,\cdot\rangle} \Leftrightarrow R \text{ has constant sectional curvature, } \kappa.$$

Outline of Proof: Assume R has constant κ. Clearly $G_{\langle\cdot,\cdot\rangle} \subset G_\mathcal{M}$ since $R = R_{\langle\cdot,\cdot\rangle}$ and by definition we have the opposite inclusion, so one implication is complete. Now we must assume $G_\mathcal{M} = G_{\langle\cdot,\cdot\rangle}$ and show R has constant κ. In order to do this we consider a 2-plane and its image under $A \in G_\mathcal{M}$, which is also a 2-plane. Using the definition of sectional curvature we are able to show that those two arbitrary planes must have the same κ. □

We also consider more general cases. It is almost always true that $G_{R_\phi} = G_\phi$ on the weak model space with $\phi \in S^2(V^*)$, but in the balanced signature case this does not necessarily hold.

Lemma 2. *Let $\phi \in S^2(V^*)$ with rank greater than or equal to 3. Then the following are true:*

1. If $A \in G_{R_\phi}$ then $A^*\phi = \pm\phi$.

2. If the signature is unbalanced then $G_{R_\phi} = G_\phi$.

Outline of Proof: We know $A \in G_{R_\phi} \Rightarrow R_\phi = A^*R_\phi = R_{A^*\phi}$. By [3] this implies that $A^*\phi = \pm\phi$. Thus A is either in G_ϕ or is a para-isometry of ϕ, which can only exist in the balanced signature case. □

The Elements of $G_\mathcal{W}$

We begin by exploring the connection between the decomposition of the model space and the structure of $G_\mathcal{W}$.

Lemma 3. *Assume $\mathcal{M} = (V, R) = (V_1, R_1) \oplus (V_2, R_2)$ with $\overline{G}_i$ as the structure group for (V_i, R_i) and $ker\{R\} = \{0\}$. Then having V_1, V_2 as invariant subspaces for all $A \in G$ is equivalent to $G \simeq \overline{G}_1 \oplus \overline{G}_2$.*

This allows us to relate the irreducible subspaces of G_ϕ viewed from a representation theory point of view to the indecomposable subspaces of $\mathcal{M}$ viewed from a linear algebra point of view. Another result describes an important decomposition of a model space for when the kernel of R is non-trivial.

Lemma 4. *Define $\overline{V} = V/kerR$ and also define $\pi : V \to \overline{V}$ to be a projection. If $\overline{R}$ is defined by $\pi^*\overline{R} = R$ as an algebraic curvature tensor on $\overline{V}$, then $(V, R) \simeq (\overline{V}, \overline{R}) \oplus (kerR, 0)$.*

Using this decomposition we can begin to see the form of elements of $G_\mathcal{W}$. Notice that the vector spaces are not invariant spaces in the following case, because $ker\{R\} \neq \{0\}$.

Theorem 1. *On some V of dimension n with $\phi \in S^2(V^*)$ of rank k notice that for any $A \in G_\phi$ written as a matrix in the orthonormal basis ordered so that the null vectors are last, we have*

$$A = \left[\begin{array}{c|c} \overline{A} & 0 \\ \hline B & C \end{array}\right]$$

where $\overline{A} \in G_{\overline{V}}$ is $k \times k$, B can be any $(n-k) \times k$ matrix, and $C \in Gl(n-k)$.

Outline of Proof: Clearly $\overline{A} \in G_{\overline{V}}$ because it only acts on $\overline{V}$ and can not even know about the other parts of V. The zero block must be there since the $ker\phi$ is invariant. The B block can be anything because this block is simply sending portions of vectors to the kernel, which does not effect anything. Finally, the C block has to be full rank only to keep $A \in Gl(n)$. □

The Form of $\overline{A}$ Related to Model Space Decomposition

We originally started researching the $R = R_\phi \pm R_\psi$ case, for $\phi, \psi \in S^2(V^*)$, but came up with a more general result concerning model space decomposition.

Theorem 2. *If a model space $(V, R) = \bigoplus_{i=1}^{n} (V_i, R_i)$ with each (V_i, R_i) indecomposable and $kerR = \{0\}$, then, for $\overline{A} \in G_{(V,R)}$ and some permutation σ we know $\overline{A} : V_i \to V_{\sigma(i)}$.*

Outline of Proof: If $\overline{A} \in G_{(V,R)}$ sends V_i to more than one other subspace non-trivially, then the subspace $\overline{A}V_i$ will decompose. Now, using $\overline{A}^{-1} \in G_{(V,R)}$ to come back to V_i will show that V_i decomposes, which is false. Also notice that $\overline{A}$ must send something to each V_i, since $\overline{A}$ is full rank. Thus we have $\overline{A} : V_i \to V_{\sigma(i)}$. □

There are several other relevant facts about $\overline{A}$. First of all, there are constrictions on the permutation σ. $\overline{A}$ is taking input meant for R_i and sending a transformed version to $R_{\sigma(i)}$, which means for each i there must exist an endomorphism B on $\mathbb{R}^{dim(V_i)}$ such that $B^*R_i = R_{\sigma(i)}$. Notice this can only happen when $dim(V_i) = dim(V_{\sigma(i)})$. Also see that there exists some exponent p for which A^p has each V_i as an invariant subspace.

Conclusion and Opportunities for Further Research

- Notice that by combining the results of Theorem 1 and Theorem 2 we have a lot of information describing the elements of a given structure group.
- We strongly believe that Lemma 1 can be generalized to include all non-degenerate bilinear forms without much change to the proof.
- This project was initiated in order to find new invariants for pseudo-Riemannian model spaces, so making the connection back to that original goal would be important.
- Theorem 2 can certainly be written entirely from the viewpoint of representation theory, and it would be valuable to make this translation and consider its ramifications in that subject.
- Considering the special case of $(V, R) = \bigoplus_{i=1}^{n} (V_i, R_{\phi_i})$ with each $\phi_i \in S^2(V^*)$ could potentially lead to interesting results that would depend on the relation between ϕ and ψ.

Acknowledgments

I would like to thank Dr. Rolland Trapp for all of his assistance in mathematics and in facilitating this research experience and I would like to especially thank Dr. Corey Dunn for sharing his mathematical expertise and for the constant encouragement and guidance over the course of this project. This research was jointly funded by NSF Grant DMS-0850959 and California State University at San Bernardino.

References

[1] Corey Dunn. A new family of curvature homogeneous pseudo-riemannian manifolds. *Rocky Mountain Journal of Mathematics*, 39(5):1443–1465, 2006.

[2] Peter B. Gilkey. *Geometric Properties of Natural Operators Defined by the Riemannian Curvature Tensor*. World Scientific, River Edge, New Jersey, 2001.

[3] Peter B. Gilkey. *The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds*. Imperial College Press, London, 2007.

[4] Friedbert Prüfer, Franco Tricerri, and Lieven Vanhecke. Curvature invariants, differential operators and local homogeneity. *Transactions of the American Mathematical Society*, 348(11):4643–4652, 1996.

Joseph Palmer
Mathematics Department
Truman State University, Kirksville MO, 63501
Email: *jap875@truman.edu*