Extremal Graph Numbers of Graphs on Few Vertices

John Kim

May 4, 2012

1 Introduction

Let H be a graph. Suppose we start with the empty graph G on n vertices and add edges to G one at a time. It is natural to ask what is the maximum number of edges we can add to G avoiding H as a subgraph. We define this number to be the extremal graph number of H on n vertices, and we denote it by $ex(n, H)$.

When H is a triangle, computing $ex(n, H)$ amounts to finding the maximum number of edges in a triangle-free graph on n vertices. This is a well-known theorem of Mantel:

Theorem 1.1 (Mantel's Theorem)**.** *Let G be a graph with n vertices and m edges. If $m > \left(\frac{n}{2}\right)^2$, then G contains a triangle.*

Mantel's Theorem essentially says that $ex(n, K_3) \approx \left(\frac{n}{2}\right)^2$. A precise formula for $ex(n, K_3)$ is given by:

$$ex(n, K_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

The extremal triangle-free graph that attains this maximum number of edges is the complete bipartite graph with half the vertices in each part. One might expect that in general, the extremal K_n-free graph is the complete n-partite graph. Indeed, Turan proved that this is true:

Theorem 1.2 (Turan's Theorem)**.** *Let G be a graph with n vertices and m edges. Let $r \geq 2$ be an integer. If $m > \frac{n^2}{2}\left(1 - \frac{1}{r}\right)$, then G contains K_{r+1} as a subgraph.*

Again, in terms of extremal graph numbers, Turan's theorem can be interpreted as $ex(n, K_{r+1}) \approx \binom{n}{2}\left(1 - \frac{1}{r}\right)$.

Extremal graph numbers are also known for paths and large trees.

Theorem 1.3 (Extremal Graph Numbers for Paths)**.** $ex(n, P_k) \approx \frac{k-1}{2}n$.

Theorem 1.4 (Extremal Graph Numbers for Large Trees)**.** *Let T be a tree on l vertices. For sufficiently large l,*

$$ex(n, T) \approx \frac{l-2}{2}n.$$

Erdos and Sos conjectured that the formula above holds for all trees:

Conjecture 1.5 (Erdos-Sos Conjecture)**.** Let T be a tree on l vertices.

$$ex(n, T) \approx \frac{l-2}{2}n.$$

Not much is known about extremal graph numbers for cycles. One notable result says that $ex(n, C_4)$ grows like $n^{\frac{3}{2}}$. In this paper, we consider trees and cycles with few vertices and compute their extremal graph numbers for small n.

For more extremal numbers on other less interesting graphs on few vertices, see the end of my maple file: extremal.txt.

2 Cycles

The extremal graph numbers for a triangle starting at $n = 1$ are

$$0, 1, 2, 4, 6, 9, 12.$$

The last number, $ex(7, C_3) = 12$ took around 250 seconds to compute, or just over 6 minutes. This aligns with the result of Mantel's Theorem.

The extremal graph numbers for C_4 starting at $n = 1$ are

$$0, 1, 3, 4, 6, 7, 9.$$

A cursory analysis suggests that $ex(n, C_4) \approx n^{\frac{3}{2}}/2$.

The extremal graph numbers for C_5 starting at $n = 1$ are

$$0, 1, 3, 6, 7, 9, 12.$$

3 Trees

The extremal graph numbers for the path with 2 edges, P_2, starting at $n = 1$ are

$$0, 1, 2, 2, 3, 3, 4, 4.$$

Avoiding paths of length two means there can only be paths of length one, so the maximum degree is 1. Hence, the maximum number of edges is $\lfloor \frac{n}{2} \rfloor$. This aligns with Theorem 1.3, which says $ex(n, P_2) \approx \frac{1}{2}n$.

The extremal graph numbers for the path with 3 edges, P_3, starting at $n = 1$ are

$$0, 1, 3, 3, 4, 6, 6.$$

This aligns with Theorem 1.3, which says $ex(n, P_2) \approx n$.

The extremal graph numbers for a claw S_3 (star with 3 edges) are

$$0, 1, 3, 4, 5, 6, 7.$$

Claw-free means that the maximum degree is 2, so the maximum number of edges in a claw-free graph on n vertices is n (the n-cycle C_n) for $n > 2$. This aligns with the Erdos-Sos conjecture for trees.

The extremal graph numbers for the tree on 5 vertices with edges $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$ are

$$0, 1, 3, 6, 6, 7.$$

There are not enough terms to say whether it conforms to the Erdos-Sos conjecture.