Sample solutions

[This is meant to illustrate appropriate level of detail. Admittedly (i) this is a judgment call and getting the balance right takes some practice, and (ii) it does require that *you* understand your solution.]

Problem. A multigraph $G = (V, E)$ is *bipartite* if there is a partition $V = X \cup Y$ so that each edge has one end in each of X, Y. As usual, Δ denotes maximum degree.

(a) Show that any bipartite multigraph G with $\Delta_G \leq d$ is contained in a *d-regular* bipartite multigraph (possibly with additional vertices).

(b) Same for bipartite *graphs* (a.k.a. *simple* graphs).

Solutions. (Assume $G = (X \cup Y, E)$ as above.)

WMA G is *balanced*, i.e. $|X| = |Y|$: if it isn't, first extend it to a balanced (bipartite) G' by adding isolated vertices (vertices of degree 0).

(a) Let $d_G(v) = d(v)$. We proceed by induction on

$$N := \sum_{x \in X}(d - d(x)) = |X|d - |E(G)| = |Y|d - |E(G)| = \sum_{y \in Y}(d - d(y)). \quad (1)$$

If $N = 0$, G is already d-regular. Otherewise, there are $x \in X$ and $y \in Y$ of degree less than d and we can add an edge joining x and y, decreasing N. (So "multi" makes this very easy.)

[Note for example: (a) this partly omitted justification for "WMA," namely, skipped commenting on the trivial points (i) you *can* get to balance by adding isolates and (ii) proving the thing for G' also proves it for G; (b) I regard (1) as not needing justification; (c) I consider the last couple lines sufficient (experience suggests some of you would feel you should say more).]

[The following is just one (simple but wasteful) way to do (b).]

(b) WMA $N \geq d - 1$. (If it isn't, add an isolated vertex to each of X, Y.)

Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, and add:

- isolated vertices $u_1, \ldots, u_N$ to X and $v_1, \ldots, v_N$ to Y;
- edges joining each x_i to $d - d(x_i)$ of the v_j's, with each v_j used exactly once, and similarly for edges between the y_i's and u_j's;

- for each $j \in [N]$, edges joining u_i to $v_i, \ldots, v_{i+d-2}$ (with subscripts interpreted mod N).

(And this does it.)

[So I'd accept—or *prefer*—"this does it" without the easy (but maybe painful) justification: it's clear the construction does what it's supposed to (right?), and I'm willing to believe you understand this if you've come up with it.]