583 PS5 Solutions

1.(a) Given δ, l, take L_0 as in Szemerédi's Theorem (so for any $L > L_0$, any $A \subseteq [L]$ of size at least δL contains an l-term A.P.). For $x \in X$ set

$$A_x = \{i \in [L] : x \in Y_i\} \quad (x \in X).$$

It's enough to show that there's some $x \in X$ for which $|A_x| \geq \delta|L|$: our choice of L then guarantees that A_x contains an A.P. $a, a+d, \ldots, a+(l-1)d$, which gives the desired Y_i's. But $\delta|L|$ is a lower bound on the *average* of the $|A_x|$'s:

$$|X|^{-1} \sum_{x \in X} |A_x| = |X|^{-1} \sum_{i=1}^{L} |Y_i| \geq |X|^{-1}|L|\delta|X| = \delta|L|.$$

(b) We show $N_0 = L_0(\delta/2, k)$ has the desired property. Let $N > N_0$, $A \subseteq [N]$ and $|A| > \delta N$. Set $X = [2N]$ and $Y_i = A + i$ for $i \in [N]$. Then $Y_i \subseteq X$ and $|Y_i| = |A| > (\delta/2)|X|$; so, since $N > N_0$, there are $a, d \in \mathbb{P}$ so that $Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset$. But for any x in this intersection, $\{x - a, x - (a+d), \ldots, x - (a + (k-1)d)\}$ is a k-term A.P. contained in A.

2. Let $X = \{A \subseteq [n] : n/2 - K\sqrt{n} < |A| < n/2 + K\sqrt{n}\}$, with the constant K chosen so $|2^{[n]} \setminus X| < \varepsilon 2^{n-1}$. Then $|S \cap X| > \varepsilon 2^{n-1}$. Let $\mathcal{C}$ be the collection of maximal chains in X. Then $|C| < 2K\sqrt{n}$ for each $C \in \mathcal{C}$, while

$$\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} |C \cap S| = \sum_{x \in S \cap X} \frac{|\{C : x \in C \in \mathcal{C}\}|}{|\mathcal{C}|} \geq \frac{|S \cap X|}{\binom{n}{\lfloor n/2 \rfloor}} > \frac{\varepsilon}{2}\sqrt{n}.$$

Thus there is a $C \in \mathcal{C}$ with

$$|C \cap S| > \frac{\varepsilon}{2}\sqrt{n} > \frac{\varepsilon}{4K}|C|,$$

and we may apply Szemerédi's Theorem to obtain (for large enough n) an arithmetic progression in C.

Alternate (using something from last semester): Let $\mathcal{C}$ be a symmetric chain decomposition of $2^{[n]}$. Since $|\mathcal{C}| = O(2^n/\sqrt{n})$, there are at most $o(2^n)$ elements of S contained in chains of length at most (say) $n^{1/4}$; so there is some $C \in \mathcal{C}$ with $|C| > n^{1/4}$ and $|S \cap C| > \varepsilon|C|/2$, etc.

3. MP: for any k, l $(\in \mathbb{P})$, $g(kl) \geq g(k)g(l)^k$. A straightforward induction then gives the stated lower bound on g.

For the MP, let $\mathcal{A}$ and $\mathcal{B}$ be (resp.) k- and l-uniform, SF-free, intersecting hypergraphs of sizes $g(k)$ and $g(l)$ and set $V(\mathcal{H}) = V(\mathcal{A}) \times V(\mathcal{B})$ and

$$[\mathcal{H} = \{\{(a_i, b) : i \in [k], b \in B_i\} : a_1, \ldots, a_k \in \mathcal{A}, B_1, \ldots, B_k \in \mathcal{B}\}$$

(and check this works).

[Less formally: we replace each vertex a of $\mathcal{H}$ by a copy, $\mathcal{B}_a$, of $\mathcal{B}$, and then each $A \in \mathcal{A}$ corresponds to $|\mathcal{B}|^k$ edges of $\mathcal{H}$ gotten by replacing each $a \in A$ by some $B_a \in \mathcal{B}_a$ (and taking their union).]

4. Let $A_0 = \{x \in A : d_B(x) > \beta n\}$. We know $|A_0| \geq (1-\varepsilon)n > (2\beta - 1)n$ (the second inequality holding because $\beta + \varepsilon \leq 1$). So A_0 contains an edge xy. But $|N_B(x) \cap N_B(y)| > (2\beta - 1)n$, so there is an edge zw in $N_B(x) \cap N_B(y)$, and $G[\{x, y, z, w\}] = K_4$.

5. Say $x_k \in X_i$ is *good* if $\min\{|\{l < k : x_l \in X_i\}|, |\{l > k : x_l \in X_i\}|\} > \varepsilon m$, define "$y_k$ good" similarly, and say k is good if both x_k and y_k are.

Observation: If k is good, $x_k \in X_i$ and $y_k \in Y_j$, then (X_i, Y_j) is (ε-)irregular.

The number of good k's is at least $(1-4\varepsilon)n$ (why?), and X_i is in an irregular pair whenever there is a good k with $x_k \in X_i$ (again, why?); so the number of X_i's in irregular pairs is at least $(1-4\varepsilon)n/m = (1-4\varepsilon)t$.

For the bonus (the answer is yes): Start with partitions $X = U_1 \cup \cdots \cup U_{t+1}$, $Y = V_1 \cup \cdots \cup V_{t+1}$, where the U_i's and V_i's are intervals written in the natural order, but with $|U_1| = |V_{t+1}| = m$ and all other blocks of size $(t-1)m/t$. Now revise to $X_1, \ldots, X_t$ and $Y_2, \ldots, Y_{t+1}$, where $X_1 = U_1$, $Y_{t+1} = V_{t+1}$; for $i \geq 2$, X_i is U_i plus m/t elements of U_{t+1}; and for $i \leq t$, Y_i is V_i plus m/t elements of V_1. (And show this works for $t > 2\varepsilon^{-2}$, the only irregular pairs being (X_i, Y_i), $2 \leq i \leq t$.)

6. First a convenient (though avoidable) observation (*why is it true?*):

if (X, Y) is ε-irregular with density d, then there are $X' \subseteq X$ and $Y' \subseteq Y$ with $d(X', Y') \notin (d - \varepsilon, d + \varepsilon)$ and $|X'| = \varepsilon |X|$, $|Y'| = \varepsilon |Y|$

(where we pretend large numbers are integers).

Lemma. If (X, Y) is ε-irregular with density d, then there are $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $|X_0| \geq \varepsilon |X|$, $|Y_0| \geq \varepsilon |Y|$ and $d(X_0, Y_0) > d + \varepsilon^3$.

(Then start with any equipartition $V = A \cup B$ and apply the lemma at most ε^{-3} times to obtain an ε-regular pair with part sizes at least $\varepsilon^{\varepsilon^{-3}} n/2$.)

Proof of lemma. Let X' Y' be as above. If $d(X', Y') > d + \varepsilon$, take $(X_0, Y_0) = (X', Y')$. If $d(X', Y') < d - \varepsilon$, then (with $\bar{X}' = X \setminus X'$, $\bar{Y}' = Y \setminus Y'$)

$$\begin{aligned} d &= \varepsilon^2 d(X', Y') + \varepsilon(1-\varepsilon)[d(X', \bar{Y}') + d(\bar{X}', Y')] + (1-\varepsilon)^2 d(\bar{X}', \bar{Y}') \\ &< \varepsilon^2 (d - \varepsilon) + \varepsilon(1-\varepsilon)[d(X', \bar{Y}') + d(\bar{X}', Y')] + (1-\varepsilon)^2 d(\bar{X}', \bar{Y}'), \quad (1) \end{aligned}$$

implying that (at least) one of the densities on the r.h.s. of (1) is greater than $[(1-\varepsilon^2)d + \varepsilon^3]/(1-\varepsilon^2) > d + \varepsilon^3$. ∎