582 PS4 Solutions

1. Suppose $n = 2k$ and $V = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Let $\sigma(x_1), \ldots, \sigma(x_k)$ be independent symmetric Bernoullis and set $\sigma(y_i) = -\sigma(x_i)$. Then $\sigma(V)$ is automatically 0, and for $H \in \mathcal{H}$, $\sigma(H) \sim S_{m(H)}$, where $m(H) = |\{i : |H \cap \{x_i, y_i\}| = 1\}| \le t$. Let $A_H = \{|\sigma(H)| > C\sqrt{t \ln t}\}$, with C TBA. Then $\mathbb{P}(A_H) < 2t^{-C^2/2}$ (by Chernoff), and the graph on vertex set $\mathcal{H}$ with

$$H \sim H' \Leftrightarrow \text{some } \{x_i, y_i\} \text{ meets both } H \text{ and } H'$$

is a Lovász graph for the A_H's with degrees less than $2t^2$. The statement now follows from the Local Lemma provided $2et^{2-C^2/2} < 1$.

2. Noting that $M^tM = qI + J$, we have, for any $\varepsilon \in \{\pm 1\}^n$,

$$n\|M\varepsilon\|_\infty^2 \ge (M\varepsilon)^t M\varepsilon = \varepsilon^t(qI + J)\varepsilon = q\varepsilon^t\varepsilon + (\textstyle\sum \varepsilon_i)^2 \ge qn.$$

(Actually the last inequality is strict, since n is odd.)

Alternate: Suppose $V = R \cup B$ is a coloring achieving discrepancy k, and choose x and y uniformly from R and B respectively. Then

$$\begin{aligned} 1 &\ge \sum_{l \in \mathcal{H}} \mathbb{P}(x, y \in l) = \sum_{l \in \mathcal{H}} \frac{|l \cap R||l \cap B|}{|R||B|} \\ &\ge (q^2 + q + 1)\frac{((q + 1 - k)/2)((q + 1 + k)/2)}{((q^2 + q + 1)/2)^2} = \frac{(q + 1)^2 - k^2}{q^2 + q + 1}, \end{aligned}$$

and the conclusion follows.

3. Modify the proof of Beck-Fiala by setting

$$\mathcal{H}^i = \{A \in \mathcal{H} : |A \cap S^i| \ge t\}.$$

(The condition in our proof was $|A \cap S^i| > t$.)

The proof goes as before *unless* we reach an i for which M^i (the $\mathcal{H}^i \times S^i$ incidence matrix) has all row and column sums exactly t. If this does happen, we can take, for each j in S^i, ε_j to be 1 if $\varepsilon_j^i \ge 0$ and -1 otherwise (and $\varepsilon \equiv \varepsilon^i$ on T^i). This gives discrepancy at most $2t - 3$ on $\mathcal{H} \setminus \mathcal{H}^i$ (as in class), and at most t on $\mathcal{H}^i$ (so we use $t \le 2t - 3$).

4. (Due to D.J. Kleitman, originally proved via shifting.) Notice that $\mathcal{A}, \mathcal{B}$ cross-intersecting is the same as

$$(\partial^{n-k-\ell}\mathcal{A}^c) \cap \mathcal{B} = \emptyset, \tag{1}$$

where $\mathcal{A}^c = \{\bar{A} : A \in \mathcal{A}\}$ (and ∂^i is ∂ applied i times). If, on the other hand, $|\mathcal{A}^c| = |\mathcal{A}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$, then KK implies $|\partial^{n-k-\ell}\mathcal{A}^c| > \binom{n-1}{(n-k)-(n-k-\ell)} = \binom{n-1}{\ell}$, in which case (1) gives $|\mathcal{B}| < \binom{n}{\ell} - \binom{n-1}{\ell} = \binom{n-1}{\ell-1}$.

5. (Induction on n.) For $\varepsilon \in \{0,1\}$, let $A_\varepsilon = \{x \in A : x_n = \varepsilon\}$ and $a_\varepsilon = |A_\varepsilon|$, and assume (w.l.o.g.) $a_0 \geq a_1$. Then

$$|\nabla(A)| \geq a_0(n-1-\log a_0) + a_1(n-1-\log a_1) + (a_0 - a_1). \tag{2}$$

(The first two terms, corresponding to edges in $\{x_n = 0\}$ and $\{x_n = 1\}$, are given by induction; the third is a lower bound on $|\nabla(A_0, \{x_n = 1\} \setminus A_1)|$.) So we want the r.h.s. of (2) to be at least $a(n - \log a)$, which we rewrite as

$$\frac{a_0}{a_0+a_1}\log\frac{a_0+a_1}{a_0} + \frac{a_1}{a_0+a_1}\log\frac{a_0+a_1}{a_1} \geq \frac{2a_1}{a_0+a_1}.$$

Equivalently, the *binary entropy* of α,

$$H(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha) \quad (\alpha \in [0,1]),$$

satisfies $H(\alpha) \geq 2\alpha$ for $\alpha \in [0, 1/2]$, which is true (e.g. look at its graph).

[Note: It's slightly easier to show the equivalent $|E(A)| \leq (1/2)a\log a$.]

6. We show that $n = R(3k, \ldots, 3k)^3$ (r copies) has the stated property (so the statement is true). Let $f : [n] \times [n] \times [n] \to [r]$ and for $1 \leq i < j < k \leq n$ set $g(\{i,j,k\}) = f(i,j,k)$. Thus $g : \binom{[n]}{3} \to [r]$ and by our choice of n there is some $D = \{i_1 < \cdots < i_{3k}\} \subseteq [n]$ with g constant on $\binom{D}{3}$. Now take $A = \{i_1, \ldots, i_k\}$, $B = \{i_{k+1}, \ldots, i_{2k}\}$ and $C = \{i_{2k+1}, \ldots, i_{3k}\}$.

7.(a) Given δ, l, take L_0 as in Szemerédi's Theorem (so for any $L > L_0$, any $A \subseteq [L]$ of size at least δL contains an l-term A.P.). For $x \in X$ set

$$A_x = \{i \in [L] : x \in Y_i\} \quad (x \in X).$$

It's enough to show that there's some $x \in X$ for which $|A_x| \geq \delta|L|$: our choice of L then guarantees that A_x contains an A.P. $a, a+d, \ldots, a+(l-1)d$, which gives the desired Y_i's. But $\delta|L|$ is a lower bound on the *average* of the $|A_x|$'s:

$$|X|^{-1} \sum_{x \in X} |A_x| = |X|^{-1} \sum_{i=1}^{L} |Y_i| \geq |X|^{-1}|L|\delta|X| = \delta|L|.$$

(b) We show $N_0 = L_0(\delta/2, k)$ has the desired property. Let $N > N_0$, $A \subseteq [N]$ and $|A| > \delta N$. Set $X = [2N]$ and $Y_i = A + i$ for $i \in [N]$. Then $Y_i \subseteq X$ and $|Y_i| = |A| > (\delta/2)|X|$; so, since $N > N_0$, there are $a, d \in \mathbb{P}$ so that $Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset$. But for any x in this intersection, $\{x - a, x - (a+d), \ldots, x - (a + (k-1)d)\}$ is a k-term A.P. contained in A.