1. Suppose $n = 2k$ and $V = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Let $\sigma(x_1), \ldots, \sigma(x_k)$ be independent symmetric Bernoullis and set $\sigma(y_i) = -\sigma(x_i)$. Then $\sigma(V)$ is automatically 0, and for $H \in \mathcal{H}$, $\sigma(H) \sim S_{m(H)}$, where $m(H) = |\{i : |H \cap \{x_i, y_i\}| = 1\}| \leq t$. Let $A_H = \{|\sigma(H)| > C\sqrt{t \ln t}\}$, with $C$ TBA. Then $P(A_H) < 2t^{1-2t^{1/2}}$ (by Chernoff), and the graph on vertex set $\mathcal{H}$ with $H \sim H' \iff$ some $\{x_i, y_i\}$ meets both $H$ and $H'$ is a Lovász graph for the $A_H$’s with degrees less than $2t^2$. The statement now follows from the Local Lemma provided $2t^{1-2t^{1/2}} < 1$.

2. Noting that $M^tM = qI + J$, we have, for any $\varepsilon \in \{\pm 1\}^n$,

$$n\|M\varepsilon\|_\infty^2 \geq (M\varepsilon)^tM\varepsilon = \varepsilon^t(qI + J)\varepsilon = q\varepsilon^t\varepsilon + (\sum \varepsilon_i)^2 \geq qn.$$  

(Actually the last inequality is strict, since $n$ is odd.)

Alternate: Suppose $V = R \cup B$ is a coloring achieving discrepancy $k$, and choose $x$ and $y$ uniformly from $R$ and $B$ respectively. Then

$$1 \geq \sum_{l \in \mathcal{H}} P(x, y \in l) = \sum_{l \in \mathcal{H}} \frac{|l \cap R||l \cap B|}{|R||B|} \geq (q^2 + q + 1) \frac{(q + 1 - k)/2)((q + 1 + k)/2)}{(q^2 + q + 1)/2)^2} = \frac{(q + 1)^2 - k^2}{q^2 + q + 1},$$

and the conclusion follows.

3. Modify the proof of Beck-Fiala by setting

$$\mathcal{H}^i = \{A \in \mathcal{H} : |A \cap S^i| \geq t\}.$$  

(The condition in our proof was $|A \cap S^i| > t$.)

The proof goes as before unless we reach an $i$ for which $M^i$ (the $\mathcal{H}^i \times S^i$ incidence matrix) has all row and column sums exactly $t$. If this does happen, we can take, for each $j$ in $S^i$, $\varepsilon_j$ to be 1 if $\varepsilon_j^i \geq 0$ and $-1$ otherwise (and $\varepsilon \equiv \varepsilon^i$ on $T^i$). This gives discrepancy at most $2t - 3$ on $\mathcal{H} \setminus \mathcal{H}^i$ (as in class), and at most $t$ on $\mathcal{H}^i$ (so we use $t \leq 2t - 3$).
4. (Due to D.J. Kleitman, originally proved via shifting.) Notice that \( \mathcal{A}, \mathcal{B} \) cross-intersecting is the same as

\[(\partial^{n-k-\ell} \mathcal{A}^c) \cap \mathcal{B} = \emptyset, \quad (1)\]

where \( \mathcal{A}^c = \{ A : A \in \mathcal{A} \} \) (and \( \partial^i \) is \( \partial \) applied \( i \) times). If, on the other hand, \(|\mathcal{A}^c| = |\mathcal{A}| > \binom{n-1}{k-1} = \binom{n-1}{n-k} \), then KK implies \(|\partial^{n-k-\ell} \mathcal{A}^c| > \binom{n-1}{(n-k)-(n-k-\ell)} = \binom{n-1}{\ell-1} \), in which case (1) gives \(|\mathcal{B}| < \binom{n}{\ell} - \binom{n-1}{\ell-1} \).

5. (Induction on \( n \).) For \( \varepsilon \in \{0, 1\} \), let \( A_\varepsilon = \{ x \in A : x_n = \varepsilon \} \) and \( a_\varepsilon = |A_\varepsilon| \), and assume (w.l.o.g.) \( a_0 \geq a_1 \). Then

\[|\nabla(A)| \geq a_0(n-1-\log a_0) + a_1(n-1-\log a_1) + (a_0-a_1) \quad (2)\]

(The first two terms, corresponding to edges in \( \{x_n = 0\} \) and \( \{x_n = 1\} \), are given by induction; the third is a lower bound on \(|\nabla(A_0, \{x_n = 1\} \setminus A_1)|\).)

So we want the r.h.s. of (2) to be at least \( a(n-\log a) \), which we rewrite as

\[
\frac{a_0}{a_0+a_1} \log \frac{a_0+a_1}{a_0} + \frac{a_1}{a_0+a_1} \log \frac{a_0+a_1}{a_1} \geq \frac{2a_1}{a_0+a_1}.
\]

Equivalently, the binary entropy of \( \alpha \),

\[H(\alpha) = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha) \quad (\alpha \in [0, 1]),\]

satisfies \( H(\alpha) \geq 2\alpha \) for \( \alpha \in [0, 1/2] \), which is true (e.g. look at its graph).

[Note: It’s slightly easier to show the equivalent \(|E(A)| \leq (1/2)a \log a\].

6. We show that \( n = R(3k, \ldots, 3k)^3 \) (\( r \) copies) has the stated property (so the statement is true). Let \( f : [n] \times [n] \times [n] \to [r] \) and for \( 1 \leq i < j < k \leq n \) set \( g(\{i, j, k\}) = f(i, j, k) \). Thus \( g : \binom{[n]}{3} \to [r] \) and by our choice of \( n \) there is some \( D = \{i_1 < \cdots < i_{3k}\} \subseteq [n] \) with \( g \) constant on \( \binom{D}{3} \). Now take \( A = \{i_1, \ldots, i_k\} \), \( B = \{i_{k+1}, \ldots, i_{2k}\} \) and \( C = \{i_{2k+1}, \ldots, i_{3k}\} \).

7.(a) Given \( \delta, l \), take \( L_0 \) as in Szemerédi’s Theorem (so for any \( L > L_0 \), any \( A \subseteq [L] \) of size at least \( \delta L \) contains an \( l \)-term A.P.). For \( x \in X \) set

\[A_x = \{ i \in [L] : x \in Y_i \} \quad (x \in X).\]
It’s enough to show that there’s some \( x \in X \) for which \( |A_x| \geq \delta|L| \): our choice of \( L \) then guarantees that \( A_x \) contains an A.P. \( a, a+d, \ldots, a+(l-1)d \), which gives the desired \( Y_i \)’s. But \( \delta|L| \) is a lower bound on the average of the \( |A_x| \)’s:

\[
|X|^{-1} \sum_{x \in X} |A_x| = |X|^{-1} \sum_{i=1}^{L} |Y_i| \geq |X|^{-1}|L|\delta|X| = \delta|L|.
\]

(b) We show \( N_0 = L_0(\delta/2, k) \) has the desired property. Let \( N > N_0 \), \( A \subseteq [N] \) and \( |A| > \delta N \). Set \( X = [2N] \) and \( Y_i = A+i \) for \( i \in [N] \). Then \( Y_i \subseteq X \) and \( |Y_i| = |A| > (\delta/2)|X| \); so, since \( N > N_0 \), there are \( a, d \in \mathbb{P} \) so that \( Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset \). But for any \( x \) in this intersection, \( \{x-a, x-(a+d), \ldots, x-(a+(k-1)d)\} \) is a \( k \)-term A.P. contained in \( A \).