582 PS3 Solutions

1. Fix R and for $i \in \{0,\ldots,t\}$, let $Z_i = \sum_{|T\cap R|=i} x_T$; so we want

$$Z_t = (-1)^t Z_0.$$

For $i \in \{0,\ldots,t-1\}$, we have

$$0 = \sum_{|U\cap R|=i} \sum_{T\supseteq U} x_T = (t-i)Z_{i+1} + (i+1)Z_i,$$

where the first equality follows from $M(t-1,t)x = \underline{0}$ (and I accept the second without justification). So $Z_{i+1} = -\frac{t-i}{i+1} Z_i$ (for each such i), yielding

$$Z_t = (-1)^t \cdot \tfrac{1}{t} \cdot \tfrac{2}{t-1} \cdot \cdots \cdot \tfrac{t}{1} \cdot Z_0 = (-1)^t Z_0.$$

2. It's ETS that for S with $|S(v)| = t\ \forall v$, we can partition $\Gamma = \Gamma_X \cup \Gamma_Y$ so

$$S(v) \cap \Gamma_X \neq \emptyset \quad \forall v \in X \quad \text{and} \quad S(v) \cap \Gamma_Y \neq \emptyset \quad \forall v \in Y. \tag{1}$$

(We can then assign each $v \in X$ a color from $S(v) \cap \Gamma_X$ and similarly for $v \in Y$.) But for a random (uniform) partition, our assumption on the $|S(v)|$'s implies that (1) fails with probability at most $n2^{-t} < 1$; so there must be a partition for which (1) holds.

3. Fix a proper coloring $\sigma : V \to [\chi]$ (where $V = V(G)$). Given $S(v)$'s of size t, let $\gamma(s)$, $s \in \Gamma$, be chosen uniformly and independently from $[\chi]$, and set

$$T = T(\gamma) = \{v \in V : \gamma^{-1}(\sigma(v)) \cap S(v) \neq \emptyset\}.$$

Then (for any γ) $G[T]$ admits an S-legal coloring. (Any coloring that assigns each $v \in T$ a color from $\gamma^{-1}(\sigma(v)) \cap S(v)$ is proper, since all vertices colored s belong to the independent set $\sigma^{-1}(\gamma(s))$.) On the other hand, $\mathbb{E}|T| = (1-(1-1/\chi)^t)n$ (since $\mathbb{P}(v \in T) = 1-(1-1/\chi)^t$ for every $v \in V$), etc.

[The conjecture mentioned is due to Albertson, Grossman and Haas, 1998.]

4. *Observation:* If $t \in \mathbb{P}$ and $\chi(H) < \chi/t$, then there is a $W \subseteq V$ with $|G[W]| \geq |W|t/2$ and W independent in H (where for graphs *size* means number of edges).

Proof. If $V_1 \cup \cdots \cup V_m$ is a coloring of H (so $H[V_i]$ is edgeless) with $m < \chi/t$, then (clearly) $\chi(G[V_i]) > t$ for some i; so the proposition in the problem says there is some $W \subseteq V_i$ with $\delta(G[W]) \geq t$ and (therefore) $|G[W]| \geq |W|t/2$. $\square$

But the probability that there is a W as in the observation (necessarily with $|W| > t$) is less than

$$\sum_{s>t} \binom{n}{s} 2^{-st/2},$$

which (check) is $o(1)$ if $t = \lfloor 2 \log n \rfloor$.

5. Assuming $q \geq mp$, let $Y_1, \ldots, Y_m$ be independent copies of X_p and $Y = \cup Y_i$. Then $Y \sim X_r$ where $r = 1 - (1-p)^m \leq q$ and

$$1 - \mu_q(\mathcal{F}) \leq 1 - \mu_r(\mathcal{F}) = \mathbb{P}(Y \notin \mathcal{F}) \leq \mathbb{P}(Y_i \notin \mathcal{F} \ \forall i) = 2^{-m}$$

(so $m = \log_2(1/\varepsilon)$ does what we want). The other direction is similar: if $q \leq p/m$ and $\mu_q(\mathcal{F}) = \delta$, then

$$1/2 = 1 - \mu_p(\mathcal{F}) \leq (1 - \mu_q(\mathcal{F}))^m = (1-\delta)^m$$

implies $-\ln 2 \leq m \ln(1-\delta) < -m\delta$ and $m < (\ln 2)/\delta$; thus $m > (\ln 2)/\varepsilon$ implies $\mu_q(\mathcal{F}) < \varepsilon$.

6. Say (X, Y) is *good* if it gives the stated conclusion. We start with G, using the notation from the proof given in the problem. The key observation is that for *any* distinct $e, f \in G$, $\mathbb{E}Z_eZ_f = 1/4$. Thus, setting $|G| = m$, we have

$$\sigma_Z^2 = \sum_e \sum_f (\mathbb{E}Z_eZ_f - \mathbb{E}Z_e\mathbb{E}Z_f) = \sum_e (\mathbb{E}Z_e^2 - \mathbb{E}^2 Z_e) = m/4$$

(where e and f run over G); so, by Chebyshev,

$$\mathbb{P}(Z \leq .49|G|) \ \leq \ \mathbb{P}(|Z - \mu_Z| \geq .01m) \ \leq \ \frac{m/4}{(.01m)^2} \ = \ \frac{2500}{|G|} \ .$$

Repeating this with H in place of G and combining gives

$$\mathbb{P}((X, Y) \text{ is good}) \geq 1 - \left[\frac{2500}{|G|} + \frac{2500}{|H|}\right],$$

which is positive if $\min\{|G|, |H|\} > 5000$ (etc.).

7. We use the suggested notation and always assume $v, w \in L$. Let X_v be the indicator of the event $\{P_v \subseteq T_p\}$ and $X = \sum X_v$ (the number of

$v \in L$ reachable from ρ in T_p). Then $Q = \{X > 0\}$ and (using the version of Chebyshev in the problem) we need $\mathbb{E}X^2 < \mu^2/\delta$ (with $\mu = \mathbb{E}X$ and δ TBA).

We have $\mathbb{E}X_v = p^n\ \forall v$, so $\mu = (rp)^n$. In addition, $\mathbb{E}X_v X_w = p^{2n-|v\wedge w|}$ and, for a given v and $i \in \{0, \dots, n\}$,

$$|\{w : |v \wedge w| = i\}| \leq r^{n-i}.$$

(The precise value is $(r-1)r^{n-i-1}$ if $i < n$ and 1 if $i = n$.) Thus

$$\begin{aligned}
\mathbb{E}X^2 &= \textstyle\sum_v \sum_w \mathbb{E}X_v X_w = \sum_v \sum_w p^{2n-|v\wedge w|} \\
&\leq r^n \textstyle\sum_{i=0}^n r^{n-i} p^{2n-i} = (rp)^{2n} \sum_{i=0}^n (rp)^{-i} \\
&< \mu^2 \textstyle\sum_{i\geq 0} (1+\varepsilon)^{-i} = (1+\varepsilon)\mu^2/\varepsilon
\end{aligned}$$

(and we take $\delta = \varepsilon/(1+\varepsilon)$).