582 PS2 Solutions

1. Since we assume no isolates WMA $|\min(P)| \leq n/2$. In what follows x's and y's are elements of P. Set

$$l(x) = \max\{l : \exists\ x_1 < \cdots < x_l = x\}$$

(the maximum size of a chain with largest element x). Let $y_1 < \cdots < y_k$ be a longest chain in P (so $k = l(y_k)$ is the largest of the $l(x)$'s). Now just notice that any $\sigma : P \to P$ satisfying

$$\sigma(x) \begin{cases} = y_1 & \text{if } l(x) = 1, \\ \in \{y_{l-1}, y_l\} & \text{if } l(x) = l \geq 2, \end{cases}$$

belongs to End(p) and that the number of such σ's is at least $2^{n/2}$.

2. If the elements of P are $x_1, \ldots, x_n$, let G be the bigraph on $\{v_1, \ldots, v_n\} \cup \{w_1, \ldots, w_n\}$ with $v_i \sim w_j$ iff $x_i < x_j$. We just need

Claim. (i) $\beta(P) \leq n - \nu(G)$, and (ii) $w(P) \geq n - \tau(G)$.

(Then König's Theorem gives $\beta(P) \leq w(P)$, which is what we want.)

For (i): With a matching M of G associate the chain partition of P generated (in the obvious way) by the relations $\{x_i < x_j : v_i w_j \in M\}$. Then x_i is maximal in its chain iff M does not cover v_i, so the number of chains is $n - |M|$ (and then take M of size $\nu(G)$).

For (ii): If $\{v_i : i \in I\} \cup \{w_j : j \in J\}$ is a (vertex) cover of G, then $\{x_k : k \in [n] \setminus (I \cup J)\}$ is an antichain of P of size at least $n - |I \cup J|$; in particular, a cover of size $\tau(G)$ gives an antichain of size at least $n - \tau(G)$.

3. We want to show that if $\mathcal{A} \subseteq 2^{[n]}$ is an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$, then $\mathcal{A}$ is one of $\mathcal{L}_1 := \binom{[n]}{\lfloor n/2 \rfloor}$, $\mathcal{L}_2 := \binom{[n]}{\lceil n/2 \rceil}$. This is immediate from the LYM proof of Sperner unless n is odd and $\mathcal{A} \subseteq \mathcal{L}_1 \cup \mathcal{L}_2$. The result is then an instance of

Real statement: If the bigraph G on $X \cup Y$ is regular (of positive degree) and connected, then its only largest independent sets are X and Y.

(To apply, notice that $\mathcal{A}$ is independent in the natural bigraph G on $\mathcal{L}_1 \cup \mathcal{L}_2$.)

Because: If I is independent with $I \cap X = J \neq \emptyset, X$, then $I \subseteq J \cup (Y \setminus N(I))$ and $|I| \leq |Y| + |J| - |N(J)| < |Y|$, with "$<$" given by the hypotheses on G.

4. This is a consequence of the theorem of Bollobás mentioned in class in connection with LYM. Let $\binom{[n]}{2} \setminus E(G) = \{A_1, \ldots, A_m\}$, and for $i \in [m]$ let C_i be the vertex set of some copy of K_s in $G + A_i$ containing A_i, and $B_i = [n] \setminus C_i$. Then $|A_i| = 2$, $|B_i| = n - s$ and

$$A_i \cap B_j = \emptyset \quad \Leftrightarrow \quad i = j$$

(*why?*); so Bollobás gives $m \le \binom{n-s+2}{2}$, which is what we want.

(To see that the bound is sharp take $E(G) = \{xy : \{x, y\} \cap [s-2] \neq \emptyset\}$.)

5. We use "general principle" from Katona's proof of EKR. Let $X = V(\Gamma_k)$ and let Y be a hexagon-free subset of X with $|Y| = \alpha_k |X|$. Let

$$\mathcal{H} = \{X_{ij} : i, j \in [2k],\ i \neq j\},$$

where $X_{ij} = \{A \in X : i \in A \subseteq [2k] \setminus \{j\}\}$.

Then $\Gamma_k[X_{ij}] \cong \Gamma_{k-1}$ (as usual, $\Gamma_k[X_{ij}]$ is the subgraph of Γ_k induced by X_{ij}) and $Y \cap X_{ij}$ is (trivially) hexagon-free; so $|Y \cap X_{ij}| \le \alpha_{k-1} |X_{ij}|$ $(\forall i, j)$. But then by the "general principle," $(\alpha_k |X| =)$ $|Y| \le \alpha_{k-1} |X|$.

6. For $i \in [n]$, set

$$S_i A = A \setminus \{i\} \quad (A \subseteq [n])$$

and, for $\mathcal{F} \subseteq 2^{[n]}$,

$$S_i \mathcal{F} = \{S_i A : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : S_i A \in \mathcal{F}\}.$$

Then $|S_i \mathcal{F}| = |\mathcal{F}|$ and we claim that (for any $\mathcal{F}$ and i)

$$\mathcal{S}(S_i \mathcal{F}) \subseteq \mathcal{S}(\mathcal{F}). \tag{1}$$

It follows (as in EKR) that WMA $S_i \mathcal{F} = \mathcal{F}$ $\forall i$, i.e. $\mathcal{F}$ is an ideal (and QED).

Proof of (1). (*Try it first.*) Suppose instead that $X \in \mathcal{S}(S_i \mathcal{F}) \setminus \mathcal{S}(\mathcal{F})$, say with $A \in \mathrm{Tr}(S_i \mathcal{F}, X) \setminus \mathrm{Tr}(\mathcal{F}, X)$, and let $B \in S_i \mathcal{F}$ satisfy $B \cap X = A$. Then (i) $B \notin \mathcal{F}$ implies $i \notin B$ and $B \cup i \in \mathcal{F}$, (ii) $i \in X$ (or $(B \cup i) \cap X = A$), and (iii) $i \notin A$ (since $A \subseteq B$); so there must be $C \in S_i \mathcal{F}$ with $C \cap X = A \cup i$. But then $i \in C \in S_i \mathcal{F}$ implies $C \setminus i \in \mathcal{F}$, a contradiction since $(C \setminus i) \cap X = A$. $\square$

[An even easier argument from Justin (briefly):

(Induction on $|X|$ with $|X| = 1$ trivial.) Let $\mathcal{F}_x = \{A \in \mathcal{F} : x \in A\}$ and $\mathcal{F}_{\bar{x}} = \{A \in \mathcal{F} : x \notin A\}$, and observe:

(i) $A \in \mathcal{S}(\mathcal{F}_x) \cup \mathcal{S}(\mathcal{F}_{\bar{x}}) \;\Rightarrow\; x \notin A$,

(ii) $A \in \mathcal{S}(\mathcal{F}_x) \cap \mathcal{S}(\mathcal{F}_{\bar{x}}) \;\Rightarrow A \cup \{x\} \in \mathcal{S}(\mathcal{F}) \setminus [\mathcal{S}(\mathcal{F}_x) \cup \mathcal{S}(\mathcal{F}_{\bar{x}})]$.

Thus (using induction for the second inequality)

$$\begin{array}{rcl} |\mathcal{S}(\mathcal{F})| & \geq & |\mathcal{S}(\mathcal{F}_x) \cup \mathcal{S}(\mathcal{F}_{\bar{x}})| + |\mathcal{S}(\mathcal{F}_x) \cap \mathcal{S}(\mathcal{F}_{\bar{x}})| \\ & = & |\mathcal{S}(\mathcal{F}_x)| + |\mathcal{S}(\mathcal{F}_{\bar{x}})| \geq |\mathcal{F}_x| + |\mathcal{F}_{\bar{x}}| = |\mathcal{F}|. \quad] \end{array}$$

7. For any $A \in \mathcal{H}$, counting in two ways gives

$$\sum_{A \neq B \in \mathcal{H}} |B \cap A| = \begin{cases} \sum_{x \in A} (d_{\mathcal{H}}(x) - 1) = |A|(d-1) \\ \\ (|\mathcal{H}| - 1)\lambda. \end{cases}$$

So $|A| = \lambda(|\mathcal{H}| - 1)/(d-1)$ for each $A \in \mathcal{H}$.

8. (Due to Frankl and Pach.) Suppose we do have μ as in the problem and let I be minimal with $\sum_{A \supseteq I} \mu_A \neq 0$. For a contradiction it's enough to show

$$\sum_{A \cap I = J} \mu_A \neq 0 \qquad \forall J \subseteq I.$$

But inclusion-exclusion and our assumption on I give

$$\sum_{A \cap I = J} \mu_A = \sum_{J \subseteq K \subseteq I} (-1)^{|K \setminus J|} \sum_{A \supseteq K} \mu_A = (-1)^{|I \setminus J|} \sum_{A \supseteq I} \mu_A \neq 0.$$