1. Answer to (b): the estimate holds for \( k = o(\ln n) \). For such a \( k \), we have

\[
|s(n, k)| = (n - 1)! \sum \left\{ \frac{1}{i_1 \cdots i_{k-1}} : 1 \leq i_1 < \cdots < i_{k-1} \leq n - 1 \right\}
\sim \frac{(n-1)!}{(k-1)!} \left( \sum_{i=1}^{n-1} \frac{1}{i} \right)^{k-1} \sim \frac{(n-1)!}{(k-1)!} \ln^{k-1} n.
\]
For the second “\( \sim \)”: you should know \( \sum_{i=1}^{n} \frac{1}{i} \sim \ln n \), which gives it for fixed \( k \); showing it’s correct iff \( k = o(\log n) \) is an exercise. The first “\( \sim \)” is similar to what we did for \( |s(n, n - k)| \): it’s ETS

\[
\sum \left\{ \frac{1}{i_1 \cdots i_{k-1}} : i_1, \ldots, i_{k-1} \in [n - 1] \text{ not all distinct} \right\} = o(\log^{k-1} n).
\]
But the l.h.s. of this is less than

\[
\binom{k-1}{2} \sum \left\{ \frac{1}{i_1 \cdots i_{k-2}} : i_j \in [n - 1] \forall j \right\} = \binom{k-1}{2} \sum_{i=1}^{n-1} i^{-2} (\sum_{i=1}^{n-1} i^{-1})^{k-2}
= O(k^2 \ln^{k-3} n),
\]
which is \( o(\ln^{k-1} n) \) for \( k = o(\log n) \).

2. Recalling the relation between \( c(n, k) \) and \( s(n, k) \), we have

\[
\sum_k c(n, k) 2^k = (-1)^n \sum_k s(n, k)(-2)^k = (-1)^n(-2)^n = (n + 1)!
\]

3. Each composition \( a = (a_1, \ldots, a_k) \) of \( n \) maps to \( \prod a_i \) “\( (1, 2) \)-compositions” of \( 2n - 1 \): represent \( a \) in the natural way as a sequence of \( n \) \( \bullet \)'s and \( k - 1 \) \( | \)'s dividing the \( \bullet \)'s into intervals of lengths \( a_1, \ldots, a_k \); then replace each \( | \) and one \( \bullet \) from each interval by a 1, and the remaining \( \bullet \)'s by 2's. In the other direction, each \( (1, 2) \)-composition of \( 2n - 1 \) is gotten as above from exactly one \( a \), via: replace the 2's by \( \bullet \)'s and 1's alternately by \( \bullet \)'s and \( | \)'s, beginning—and ending, since the number of 1's is odd—with \( \bullet \)'s.

4. Check: (i) \( \prod_{j \geq 0} (1 + x^{2^j})^{-1} = 1 - x \), and (ii) the coeff of \( x^n \) on the l.h.s. is \( p_e(n) - p_o(n) \), where \( p_e \) (resp. \( p_o \)) means number of partitions into an even (resp. odd) number of 2-powers.
5. Observation: if \( h(x, y) = \sum_{n,k} c_{n,k} x^ny^k \) then
\[
\frac{\partial}{\partial y} h(x, y)|_{y=1} = \sum_{n} \sum_{k} kc_{n,k} x^n. \tag{1}
\]
Now let \( a_{n,k} = |\{ \lambda + n : u(\lambda) = k \}|, \ b_{n,k} = |\{ \lambda + n : v(\lambda) = k \}|, \)
\( f(x, y) = \sum_{n,k} a_{n,k} x^ny^k \) and \( g(x, y) = \sum_{n,k} b_{n,k} x^ny^k, \) and check that
\[ f(x, y) = (1 - xy)^{-1} \prod_{i \geq 2} (1 - x^i)^{-1} \]
and
\[ g(x, y) = \prod_{i \geq 1} \left( \frac{1 - x^i + yx^i}{1 - x^i} \right), \]
whence
\[
\frac{\partial}{\partial y} f(x, y)|_{y=1} = \frac{x}{(1-xy)^2} \prod_{i \geq 2} (1 - x^i)^{-1}|_{y=1} = \frac{x}{1-x} \prod_{i \geq 1} (1 - x^i)^{-1} \tag{2}
\]
and
\[
\frac{\partial}{\partial y} g(x, y)|_{y=1} = \sum_{i \geq 1} x^i \prod_{j \geq 1} (1 - x^j)^{-1} = \text{r.h.s. of (2)}. \]
But then (1) gives (with \( c_n \) meaning coefficient of \( x^n \))
\[
\sum_{\lambda \vdash n} u(\lambda) = c_n \frac{\partial}{\partial y} f(x, y)|_{y=1} = c_n \frac{\partial}{\partial y} g(x, y)|_{y=1} = \sum_{\lambda \vdash n} v(\lambda).
\]

6. For \( I \subseteq [k] \) let \( A_I = \{ E \in \binom{[n]}{k} : E \cap V_i = \emptyset \ \forall i \in I \} \) (so \( A_I = \cap_{i \in I} A_i \),
where \( A_i := A_{\{i\}} \)). Then \( \cap \overline{A}_i = \{ \text{transversals} \} \) and
\[ n^k = |h(\cap \overline{A}_i)| = \left| \sum_{I \subseteq [k]} (-1)^{|I|} h(A_I) \right| \leq 2^k \max_I |h(A_I)|, \]
which, since \( h(\cap A_I) = h(\cup_{i \in I} V_i) \), gives the desired result with \( c_k = 2^{-k} \).

7.(a) Given \( A \) and \( A_i \)'s as in the problem, the displayed sum is the probability that \textit{exactly} those \( A_i \)'s indexed by \( I \) occur. To see this, set \( B_j = A_{I \cup \{j\}} \)
(= \( A_I \cap A_j \)) for \( j \in [n] \setminus I \) (so \( B_J = A_{I \cup J} \) for \( J \subseteq [n] \setminus I \)). Then I-E gives
\[
\mathbb{P}(A_I \setminus \cup_{j \in [n] \setminus I} A_j) = \mathbb{P}(A_I \setminus \cup_{j \in [n] \setminus I} B_j) = \sum_{J \subseteq [n] \setminus I} (-1)^{|J|} \mathbb{P}(B_J) = \sum_{K \supseteq I} (-1)^{|K \setminus I|} p_K.
\]
In particular the r.h.s. is nonnegative (and \( p_0 = \mathbb{P}(A) = 1 \)).
(b) Given $p_I$'s satisfying (1) and (2), let $A = \{x_J : J \subseteq [n]\}$ ($x_J$'s distinct), set $A_i = \{x_J : J \ni i\}$ ($i \in [n]$), and define a measure $f$ on $A$ by

$$f(x_J) = \sum_{K \supseteq J} (-1)^{|K\setminus J|} p_K \geq 0.$$ 

Then (with our usual notation) $A_I = \{x_J : J \supseteq I\}$ and

$$f(A_I) = \sum_{J \supseteq I} \sum_{K \supseteq J} (-1)^{|K\setminus J|} p_K = \sum_{K \supseteq I} p_K \sum_{I \subseteq J \subseteq K} (-1)^{|K\setminus J|} = p_I.$$

In particular $\sum f(x_I) = f(A) = p_\emptyset = 1$. Thus the $f(x_I)$'s are nonnegative and sum to 1, so $f$ is a probability measure on $A$. 

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