582 PS1 Solutions

1. This becomes more or less trivial if we think of starting at the other end, i.e. with $\Pi_n$, since the number of ways to move from $\Pi_i$ to $\Pi_{i-1}$ is *always* $\binom{i}{2}$ (namely, we can combine any two of the $i$ blocks of $\Pi_i$). Thus the number of sequences is $\prod_{i=2}^n \binom{i}{2}$ $(= (n!)^2/(n2^{n-1}))$.

2. For $k \in \{0,\dots,n\}$, $\binom{x+k}{k} = \binom{x+k}{x}$ is the number of $(x+1)$-element subsets of $[x+n+1]$ with largest element $x+k+1$ (so the sum is $\binom{x+n+1}{x+1} = \binom{x+n+1}{n}$).

3. With $a_i := |\{j : x_i = \max(A_j)\}|$, the key observation is

$$\underline{v} \text{ is determined by } (a_1,\dots,a_n). \tag{1}$$

This is enough: $a_1,\dots,a_n$ is a weak $n$-composition of $m$, so the number of possibilities for the $a_i$'s is at most $\binom{m+n-1}{n-1}$. (Of course it can be less, since weak compositions needn't correspond to legal $\underline{v}$'s.)

For (1) (one way) we proceed by induction on $n$, with $n = 1$ trivial. Choose $i$ with $a_i = |\{j : x_i \in A_j\}|$, i.e. with $\max(A_j) = x_i$ for each $A_j$ containing $x_i$ (which is true, e.g., if $x_i$ is maximum under $\prec$). Now delete $x_i$ and all $A_j$'s containing it and say induction. □

4. Let $X_I = \{x \in X : x \in A_i \Leftrightarrow i \in I\}$ and let $Y_I \subseteq Y$ be the analogue for the $B$'s. Inclusion-exclusion applied to the universe $A_I$ and subsets $A_{I\cup\{j\}}$ $(j \in [m] \setminus I)$ gives

$$|X_I| = \textstyle\sum_{J\supseteq I} (-1)^{|J\setminus I|} |A_J|.$$

(and similarly for the $Y_I$'s). Combining this with our hypothesis we find that, for each $I$,

$$|X_I| = \textstyle\sum_{J\supseteq I} (-1)^{|J\setminus I|} |A_J| \neq \sum_{J\supseteq I} (-1)^{|J\setminus I|} |B_J| = |Y_I|.$$

It follows that either at least half of the $X_I$'s or at least half of the $Y_I$'s are nonempty, so $|X| \geq 2^{m-1}$.

[For the exercise: Fix $\varepsilon, \delta \in \{0,1\}^m$ with even and odd weight respectively, let $X$ be the set of even weight strings from $\{0,1\}^m$ and set

$$A_i = \{x \in X : x_i = \varepsilon_i\}, \quad B_i = \{x \in X : x_i = \delta_i\}.$$

Then $|A_I| = |B_I| = 2^{m-1-|I|}$ if $I \neq [m]$, but $A_{[m]} = \{\varepsilon\}$ and $B_{[m]} = \emptyset$.]

5. Let $A = \binom{V}{k}$ and for $i \in [k]$ let $A_i = \{E \in A : E \cap V_i = \emptyset\}$. Then $A \setminus \cup A_i = \{\text{transversals}\}$ and

$$n^k = |h(A \setminus \cup A_i)| = |\sum_{I \subseteq [k]} (-1)^{|I|} h(A_I)| \leq 2^k \max_I |h(A_I)|,$$

which, since $h(A_I) = \overline{h}(\cup_{i \notin I} V_i)$, gives the desired result with $c_k = 2^{-k}$.

6. We have $\mathbb{P}(X = 0) = \prod(1 - p_i) \to e^{-\mu}$ and

$$\mathbb{P}(X = 1) = \prod(1 - p_i) \sum \tfrac{p_i}{1-p_i} \to \mu e^{-\mu},$$

whence

$$\sum p_i/(1 - p_i) \to \mu \tag{2}$$

and

$$\prod \left[(1 - p_i) e^{p_i/(1-p_i)}\right] \to 1. \tag{3}$$

Since $(1 - x)e^{x/(1-x)}$ is 1 when $x = 0$ and is strictly increasing on $[0, 1]$, (3) implies $\max p_i \to 0$, and then (2) gives $\sum p_i \to \mu$.

[Alternate: with $q_i = p_i/(1 - p_i)$, $\mathbb{P}(X = 2) \to e^{-\mu}\mu^2/2$ gives $\sum_{i<j} q_i q_j \to \mu^2/2$, which with (2) gives $\sum q_i^2 \to 0$, $\max q_i \to 0$, etc.]