1. This becomes more or less trivial if we think of starting at the other end, i.e. with \( \Pi_n \), since the number of ways to move from \( \Pi_i \) to \( \Pi_{i-1} \) is always \( \binom{i}{2} \) (namely, we can combine any two of the \( i \) blocks of \( \Pi_i \)). Thus the number of sequences is \( \prod_{i=2}^{n} \binom{i}{2} \left( = \frac{(n!)^2}{(n2^n - 1)} \right) \).

2. For \( k \in \{0, \ldots, n\} \), \( \binom{x+k}{k} \) is the number of \( (x+1) \)-element subsets of \([x+n+1]\) with largest element \( x+k+1 \) (so the sum is \( \binom{x+n+1}{x+1} = \binom{x+n+1}{n} \)).

3. With \( a_i := |\{j : x_i = \max(A_j)\}| \), the key observation is

\[ v \text{ is determined by } (a_1, \ldots, a_n). \quad (1) \]

This is enough: \( a_1, \ldots, a_n \) is a weak \( n \)-composition of \( m \) (right?), so the number of possibilities for the \( a_i \)'s is at most \( \binom{m+n-1}{n-1} \) (“at most” since it may be that some choices of \( a_1, \ldots, a_n \) don’t correspond to possible \( v \)'s).

For (1) (one way) we proceed by induction on \( n \), with \( n = 1 \) trivial. Choose \( i \) with \( a_i = |\{j : x_i \in A_j\}| \), i.e. with \( \max(A_j) = x_i \) for each \( A_j \) containing \( x_i \) (which is true, e.g., if \( x_i \) is maximum under \( \prec \)). Now delete \( x_i \) and all \( A_j \)'s containing it and say induction.

4. The probability is \( \frac{(n-k)!}{(n-k)! n!} = \frac{\binom{n-k}{k}}{\binom{n}{k}} \). Sample ways to show this is \( \sim e^{-1} \):

(A) rewrite in terms of factorials and use Stirling:

\[
\frac{(n-k)!}{(n-k)! n!} \sim \left[ \frac{(n-k)^{2(n-k)}}{(n-2k)^{n-2k}} \right]^{1/2} \frac{(n-k)^{2(n-k)}}{(n-2k)^{n-2k} n^n}
\]

(with some cancellation of \( 2\pi \)'s and powers of \( e \)). The first expression is asymptotically 1. We can rewrite the second as, for example,

\[
\left[ \frac{(n-k)^{2(n-k)}}{n(n-2k)} \right]^{n} \left[ \frac{n-2k}{n-k} \right]^{2k} \left[ 1 + \frac{k^2}{n^2-2kn} \right] \left[ 1 - \frac{k}{n-k} \right]^{2k} \sim e \cdot e^{-2} = e^{-1}.
\]

For “\( \sim \)” we use \( k \sim \sqrt{n} \) and two applications of \( e^{x-x^2} < 1 + x < e^x \). (In each case the contribution of the “\( x^2 \)” term is negligible.)

(B) rewrite

\[
\frac{(n-k)k}{(n)k} = \frac{(n-k)k}{(n-k)k} \frac{n^k}{(n)k} \frac{n-k}{n}^k
\]
and (again using $e^{x-x^2} < 1 + x < e^x$ and $k \sim \sqrt{n}$) find that the factors on the r.h.s. are asymptotically $e^{-1/2}$, $e^{1/2}$ and $e^{-1}$ respectively.

(C) Since $(n - k - i)/(n - i)$ is decreasing in $i$, we have

$$(1 - \frac{k}{n-k})^k < \frac{(n-k)_k}{(n)_k} < (1 - \frac{k}{n})^k$$

(and use $e^{x-x^2} < 1 + x < e^x$ and $k \sim \sqrt{n}$ to show both bounds are $\sim e^{-1}$).

5. Define events $A_m = \{0 \not\in \{X_1, \ldots, X_m\}\}$. Then (main point, I guess) $P(\cap A_m) = 0$ (why?). So monotone convergence says $P(A_m) \to 0$, and the statement in the problem follows since $Q_m \subseteq A_m$. 