582 Solutions to "suggested problems"

Here are examples, for a couple of the "suggested problems," of what I'd consider acceptable solutions. In general I'll try to avoid giving problems that require long, prosy explanations. So if you do have the solution, it shouldn't take too much space to write down (unless I warn you otherwise); and if you don't, please don't try to talk the problem to death. As indicated in a couple places below, you don't have to spell out things that are really obvious. (Admittedly "obvious" is a matter of opinion, but it's important to develop a feel for what needs to be explained.)

3.(a) The answer is $(k+1)^n$ because the correspondence $(T_1,\ldots,T_k)\mapsto f$ with

$$f(i)=\begin{cases}\min\{j: i\in T_j\} & \text{if } i\in\cup T_j\\ k+1 & \text{otherwise}\end{cases}$$

gives a bijection between the set of sequences described in the problem and the set of functions $[k+1]^{[n]}$.

[The point here was to recognize the correspondence. Once you do, the rest is pretty obvious and doesn't need to be justified further.]

(b) For $\mathcal{A}=(A_1,\ldots,A_k)\in S$ and $i\in[n]$, write $\mathcal{A}\succ i$ for "$i\in\cup_{j=1}^k A_j$." Note $|S|=2^{nk}$ and $|\{\mathcal{A}\in S:\mathcal{A}\not\succ i\}|=2^{(n-1)k}$. Thus

$$\sum_{\mathcal{A}\in S}|A_1\cup\cdots\cup A_k|=\sum_{i\in[n]}|\{\mathcal{A}\in S:\mathcal{A}\succ i\}|=n\left(2^{nk}-2^{(n-1)k}\right).$$

[Note you don't need to justify the assertions in the second sentence.]

5. Answer: $S(n,k)\sim k^n/k!$, or equivalently,

$$k!S(n,k)\sim k^n. \tag{1}$$

Proof. Write $g(n,k)$ for the number of *nonsurjective* maps from $[n]$ to $[k]$. Then $k!S(n,k)=k^n-g(n,k)$, so for (1) it's ETS $g(n,k)=o(k^n)$.

[Note no further justification required for "Then" and "ETS."]

But

$$g(n,k)\le\sum_{i\in[k]}|([k]\setminus\{i\})^{[n]}|=k(k-1)^n=o(k^n)$$

(e.g. since $k(k-1)^n/k^n<ke^{-n/k}\to 0$).

□

Remark. This actually gives (1) provided

$$k < \frac{n}{\log n - \log\log n + \omega(1)}.$$

It's also true that (1) does *not* hold for larger k ($k > \frac{n}{\log n - \log\log n + O(1)}$), but showing this is a bit harder.