1. Show that Frankl’s (union-closed families) conjecture implies the theorem of Reimer proved in class.

2. A possible strengthening of Frankl’s conjecture: If \( f : 2^{[n]} \to \mathbb{R}^+ \) satisfies
\[
f(A \cup B) \geq \min\{f(A), f(B)\} \quad \forall A, B \subseteq [n]
\]
and, say (to avoid stupidities), \( f([n]) \geq f(\emptyset) \), then there is an \( x \in [n] \) with \( \sum \{f(A) : A \ni x\} \geq \frac{1}{2} \sum f(A) \). Show this is (unfortunately) not true.
[Note: The example I have in mind (for which, BTW, there’s just one \( A \) with \( f(A) \not\in \{0, 1\} \)) isn’t huge, but it’s also not very small. (I didn’t try to figure out how small one could make it.)]

3. (a) Prove the following statement. [This was conjectured by Alon-Seymour-Thomas and proved (independently) by A. Kotlov and G. Tardos, but is not hard despite the pedigree.]

If \( X \) is a set of more than half the vertices of \( \{0, 1\}^n \), then some component of the subgraph (of the Hamming graph) induced by \( X \) meets every hyperplane \( \{x_i = \varepsilon\} \) \( i \in [n], \varepsilon \in \{0, 1\} \).
[Note the lower bound on \( |X| \) can’t be decreased. Hint: a funny shift.]
(b) Show that \( |\nabla^- (\mathcal{G})| \leq 2^{n-1} \) for any simply rooted \( \mathcal{G} \subseteq 2^{[n]} \).
[This was the reason for (a), but it would also be interesting to see if one can do it without (a).]

4. (Repeating from class:)
(a) If \( \mathcal{F} \) is \( s \)-saturated \( (s \geq 3) \), then it contains a maximal intersecting family.
(b) Is it true that if \( \mathcal{F} \) is \( (s + 1) \)-saturated then it contains an \( s \)-saturated family? (I don’t know, but guess NO.)
5. [An event \( E \subseteq \Omega = \{0, 1\}^S \) is determined by \( I \subseteq S \) if membership of \( x \) in \( E \) depends only on \( x_I \).] Let \( Q = \bigcup_{i \in I} Q_i \) (\( I \) some index set), where each \( Q_i \subseteq \Omega \) is determined by at most \( u \) variables. Let \( \mu \) be a product probability measure on \( \Omega \) and assume \( \xi : 2^T \to \mathbb{R} \) is Lipschitz (i.e. \( x \sim y \Rightarrow |\xi(x) - \xi(y)| \leq 1 \)). Show that (for any \( t \))

\[
\mu(\xi \geq t|Q) \leq \mu(\xi + u \geq t).
\]

6. [Let \( \mu_n \) be uniform measure on \( \Omega_n = 2^{E(K_n)} \equiv \{ \text{graphs on } [n] \} \].] (Repeating from class:) Show that if \( H \) is a star forest then for any \( \varepsilon > 0 \) there is (for some \( n \)) an \( H \)-intersecting family \( \mathcal{F} \subseteq \Omega_n \) with \( \mu_n(\mathcal{F}) > 1/2 - \varepsilon \).

7. [Mostly opens, with one actual problem at “show.”]

(a) For \( X \subseteq [n] \) and \( i \in [n] \), let \( X + i = \{x + i : x \in X\} \), where addition is modulo \( n \). Here are two old conjectures (the first, which is reminiscent of Simonovits-Sós, is due to Chung, Frankl, Graham and Shearer; the second, due to Griggs and Walker, was motivated by the first):

**Conjecture 1.** Let \( X \) be a \( k \)-subset of \([n]\) and suppose \( \mathcal{F} \subseteq 2^{[n]} \) satisfies

\[
\forall A, B \in \mathcal{F}, \ A \cap B \supseteq X + i \quad \text{for some } i \in [n].
\]

Then \( |\mathcal{F}| \leq 2^{n-k} \).

**Conjecture 2.** For any \( X \subseteq [n] \) of size \( k \), there is a \( k \times n \) \( \mathbb{Z}_2 \)-matrix \( M \) such that the columns of \( M \) indexed by \( X + i \) are linearly independent for each \( i \in [n] \).

Show Conjecture 2 implies Conjecture 1.

(b) Another way to say Conjecture 2:

**Conjecture 2’.** If \( X \) is a \( k \)-subset of \([n]\) and for \( i \in [n] \), \( A_i = X + i \) (addition again modulo \( n \)), then there are \( v_1, \ldots, v_n \in \mathbb{Z}_2^k \) such that

\[
\{v_j : j \in A_i\} \text{ is independent for every } i.
\]  

Now wandering a bit, here are two conjectures of a similar flavor (due to R. Meshulam and JK; *you shouldn’t assume they’re right*), in which \( A_1, \ldots, A_n \) are again \( k \)-subsets of \([n]\):

**Conjecture 3.** For any \( A_1, \ldots, A_n \), there are \( v_1, \ldots, v_n \) spanning \( \mathbb{Z}_2^k \) such that for each \( i \), \( v_i \) is spanned by \( \{v_j : j \in A_i\} \).
Conjecture 4. If for each $t$ and $1 \leq i_1 < \cdots < i_t \leq n$,
\[
|A_{i_1} \cap \cdots \cap A_{i_t}| \leq \max\{k - t + 1, 0\} \quad \text{and} \\
|A_{i_1} \cup \cdots \cup A_{i_t}| \geq \min\{k + t - 1, n\},
\]
then there exist $v_1, \ldots, v_n \in \mathbb{Z}_2^k$ satisfying (1).

8. For any $B \subseteq 2^X$,
\[
\max\{|F| : F \subseteq 2^X, B\text{-intersecting}\} = \max\{|F| : F \subseteq 2^X, B\text{-agreeing}\}.
\]
[Definitions (you already know): $F$ is $B$-intersecting (resp. $B$-agreeing) if for any $F, G \in F$ there’s a $B \in B$ with $F \cap G \supseteq B$ (resp. $F, G$ agree on $B$).]

9. Prove the Triangle Removal Lemma: For each $\varepsilon > 0$ there is a $\delta > 0$ such that (for any $n$) any $n$-vertex graph $G$ containing at most $\delta n^3$ triangles can be made triangle-free by removal of at most $\varepsilon n^2$ edges.

[Some of you have seen this in 583. The TRL is a seminal result of Ruzsa and Szemerédi that (unexpectedly) turned out to imply Roth’s Theorem on 3-term arithmetic progressions (the first case of Szemerédi’s Theorem). Here it’s intended as a Regularity Lemma exercise (similar to what we did in class for EKR, except now using the Counting Lemma for real). The true relation between $\varepsilon$ and $\delta$ is a major open problem (see e.g. Conlon-Fox survey from 2013); a significant improvement over what one gets from the RL was given by Fox in 2011, but the situation remains pathetic.]

10. For a graph $G$ and $H$ the bipartite double cover of $G$ (that is,
\[
V(H) = \{v' : v \in V(G)\} \cup \{v'' : v \in V(G)\}
\]
with all these vertices distinct] and $E(H) = \{v'w'' : vw \in E(G)\}$, show
\[
i(H) \geq i^2(G).
\]

Then use the fact that
\[
i(G) \leq (2^{d+1} - 1)^{N/(2d)}
\]
for $d$-regular, $N$-vertex bipartite $G$ (for all $N$ and $d$) to show that the same bound holds for general ($d$-regular, $N$-vertex) $G$.  

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11. Prove that (as mentioned in class) the number of sum-free subsets of \([n]\) is less than \(2^{(1+o(1))n/2}\).

[Hints/suggestions:
(a) Use the “nearly-regular” form of Alon’s independent set bound, applied to each member of some collection of graphs on \([n]\).
(b) (Minor:) We only need to worry about slightly large (sum-free) sets.
(c) Try it first in \(\mathbb{Z}_n\) (this is easier, right?). The proof for \([n]\) is similar but requires one clever (simple, but a nice challenge to come up with) idea.]

12. Show that (as mentioned in class) \(\text{ex}(n,C_4) \sim n^{3/2}\).
[Again as said in class: the upper bound is elementary; for the lower bound you may want to recall finite projective planes.]

13.⋆ For a graph \(G\) on \(V\) and positive integer \(t\), let \(\lambda_t(G)\) be the largest \(\alpha \in [0,1]\) such that for every \(L = (L_v : v \in V)\) with \(|L_v| = t \forall v\), there is an \(L\)-coloring of some \(\alpha|V|\) vertices of \(G\). Then for \(t \leq s\) let

\[
\lambda_{s,t} = \inf \{\lambda_t(G) : \chi_s(G) = s\}.
\]

Conjecture (Albertson, Grossman & Haas, 1998): For every \(t \leq s\), \(\lambda_{s,t} = t/s\).

The analogous statement for \(\chi\) is of course trivial, as is \(\lambda_{s,t} \leq t/s\) (why?). You might convince yourself that the conjecture is at least true if \(t|s\).

(a) Show that for any \(G\) and \(t\),

\[
\lambda_t(G) \geq 1 - (1 - 1/\chi(G))^t.
\]

(b) Show (for \(t < s\)) that \(\lambda_{s,t} \geq q_{s,t}\), where \(q_{s,t}\) is the unique positive root of

\[
1 - x - (1 - \frac{1-x}{s-t})^t = 0.
\]

[Both are nice exercises—especially (b), though the result isn’t so appealing. You’ll want to do something probabilistic (of course), but nothing fancy. Having done the easier (a) may provide a clue for (b).]
14. Recall “Fact 2” from class (let’s say just for $r = 2$):

For each $\delta > 0$ there is an $\varepsilon > 0$ such that any $H \subseteq K_n$ satisfying $|H| > (1 - \varepsilon)n^2/4$ and $\tau(H) < \varepsilon n^3$ is $(\delta n^2)$-close to bipartite

(where $\tau$ is number of triangles). Prove this without using the Triangle Removal Lemma (Problem 9).

[Not hard but AFAIK not completely easy to find (but I’d be happy to hear otherwise).]

15. Use Theorems D3 and S3 to show existence of $K_4$-free graphs $G$ satisfying $G \rightarrow (K_3)_2$. 