642.583 Problem Set 5 (final)

1.(a) For all $\delta > 0$ and $l \in \mathbb{P}$ there is an $L_0 = L_0(\delta, l)$ for which the following is true. If $L > L_0$, X is a finite set, and $Y_1, \ldots, Y_L \subseteq X$ satisfy

$$|Y_i| \geq \delta |X| \quad \forall i \in [L],$$

then there are $a, d \in \mathbb{P}$ such that ($a + (l-1)d \leq L$ and)

$$Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset.$$

(b) Show that the statement in (a) *implies* Szemerédi's Theorem (in the form: for all $\delta > 0$ and $k \in \mathbb{P}$ there is an N_0 such that for any $N > N_0$, any $A \subseteq [N]$ of size at least δN contains a k-term A.P.).

2. (Here SF means 3-sunflower.) Let $f(n) = f_3(n)$ be the maximum size of an n-uniform, SF-free $\mathcal{H}$ and let $g(n)$ be the maximum size of such an $\mathcal{H}$ that's also *intersecting*.

Fact: $g(n) \geq 10$.

[*Reason*: Form $\mathcal{H}$ by identifying antipodal points of an icosahedron.]

Use the Fact to show that if n is a power of 3, then $g(n) \geq 10^{(n-1)/2}$.

[So (the point of the problem) $f(n) \geq 2g(n)$ is (much) larger than 2^n. If you find the construction: (a) please make some effort to find a nice way to describe it; (b) you can go light on verification details.]

4. Let $G = (V, E)$, $V = A \sqcup B$, with (A, B) ε-regular and $|A| = |B| = n$. Assume further that $d(A, B) \geq \beta + \varepsilon$ with $\beta > 1/2$ (and $\varepsilon > 0$), and that $\alpha(G) \leq (2\beta - 1)n$. Then $\omega(G) \geq 4$. (Recall α and ω are independence and clique number respectively.)

5. Let G be bipartite on $(X = \{x_1, \ldots, x_n\}) \cup (Y = \{y_1, \ldots, y_n\})$ with $x_i \sim y_j$ iff $i \leq j$. (This is sometimes called a *half-graph*.)

Suppose $\{X_i\}_{i=1}^t$ and $\{Y_i\}_{i=1}^t$ are partitions of X and Y with $|X_i| = |Y_i| = n/t =: m$ $\forall i$. Show that if $\varepsilon > 0$ is slightly small, then the number of ε-irregular pairs (X_i, Y_j) isn't much less than t. (E.g. if $\varepsilon = o(1)$ then the number is at least $(1 - o(1))t$.)

[Not part of the problem but you could try: Can it be less than t?]