642.582 Problem Set 4 (final installment)

1. Show that there is a fixed C such that if $\mathcal{H}$ is a t-uniform, t-regular hypergraph on $V = [n]$, with n even and $t > 1$, then there is an $f : V \to \{\pm 1\}$ with

$$|f(H)| \le C\sqrt{t \ln t} \quad \forall H \in \mathcal{H} \tag{1}$$

and

$$f(V) = 0. \tag{2}$$

[Easy once found ...]

2. A *projective plane of order* q is a hypergraph $\mathcal{H}$ on vertex set V (elements of V and $\mathcal{H}$ are usually called *points* and *lines*) satisfying:

(i) $|\mathcal{H}| = |V| = q^2 + q + 1$ $(=: n$, say);

(ii) $|l| = q + 1 = d_{\mathcal{H}}(x)$ $\forall l \in \mathcal{H}$ and $x \in V$;

(iii) $\forall x, y \in V$ $(x \ne y)$ $\exists!\ l \in \mathcal{H}$ with $x, y \in l$; and

(iv) $\forall l, m \in \mathcal{H}$ $(l \ne m)$ $\exists!\ x \in l \cap m$.

(The conditions (i)-(iv) are somewhat redundant. As mentioned in class, you can find more on projective planes in vLW.)

Show that if $\mathcal{H}$ is a projective plane of order q, then $\mathrm{disc}(\mathcal{H}) \ge \sqrt{q}$.

[Pseudo-hint: can be done in essentially one line.]

3. As in class, let

$$f(t) = \max\{\mathrm{disc}(\mathcal{H}) : \Delta_{\mathcal{H}} = t\}$$

(where Δ is maximum degree). Show $f(t) \le 2t - 3$ for $t \ge 3$.

[It's okay to refer to, rather than repeat, items from class.]

4. Say $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are *cross-intersecting* if

$$A \cap B \ne \emptyset \quad \forall A \in \mathcal{A},\ B \in \mathcal{B}.$$

Use Kruskal-Katona to show that if $k+l \le n$ $(k, l \ge 1)$ and $\mathcal{A} \subseteq \binom{n}{k}$, $\mathcal{B} \subseteq \binom{n}{l}$ are cross-intersecting, then either $|\mathcal{A}| \le \binom{n-1}{k-1}$ or $|\mathcal{B}| < \binom{n-1}{l-1}$.

[Another way (but *not here please*): shifting as in the proof of EKR.]

5. Show that for $A \subseteq Q_n$ of size a (and $\log = \log_2$),

$$|\nabla(A)| \geq a(n - \log a).$$

[Your proof should *not* give (or depend on) the precise result stated in class.]

6. Prove or disprove: For all $k, r \in \mathbb{P}$ there is an n such that for any $f : [n] \times [n] \times [n] \to [r]$ there are $A, B, C \subseteq [n]$ with $|A| = |B| = |C| = k$ and f constant on $A \times B \times C$.

[Note all problem *parts* have the same weight.]

7.(a) For all $\delta > 0$ and $l \in \mathbb{P}$ there is an $L_0 = L_0(\delta, l)$ for which the following is true. If $L > L_0$, X is a finite set and $Y_1, \ldots, Y_L \subseteq X$ satisfy

$$|Y_i| \geq \delta |X| \quad \forall i \in [L],$$

then there are $a, d \in \mathbb{P}$ such that ($a + (l-1)d \leq L$ and)

$$Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset.$$

(b) Show that the statement in (a) *implies* Szemerédi's Theorem (in the form: for all $\delta > 0$ and $k \in \mathbb{P}$ there is an N_0 such that for any $N > N_0$, any $A \subseteq [N]$ of size at least δN contains a k-term A.P.).