1. Show that there is a fixed $C$ such that if $\mathcal{H}$ is a $t$-uniform, $t$-regular hypergraph on $V = [n]$, with $n$ even and $t > 1$, then there is an $f : V \to \{\pm 1\}$ with

$$|f(H)| \leq C\sqrt{t\ln t} \quad \forall H \in \mathcal{H}$$

and

$$f(V) = 0.$$  \hspace{1cm} (1)

[Easy once found ... ]

2. A projective plane of order $q$ is a hypergraph $\mathcal{H}$ on vertex set $V$ (elements of $V$ and $\mathcal{H}$ are usually called points and lines) satisfying:

(i) $|\mathcal{H}| = |V| = q^2 + q + 1$ (=: $n$, say);

(ii) $|l| = q + 1 = d_H(x) \quad \forall l \in \mathcal{H}$ and $x \in V$;

(iii) $\forall x, y \in V \ (x \neq y) \exists! \ l \in \mathcal{H}$ with $x, y \in l$; and

(iv) $\forall l, m \in \mathcal{H} \ (l \neq m) \exists! \ x \in l \cap m$.

(The conditions (i)-(iv) are somewhat redundant. As mentioned in class, you can find more on projective planes in vLW.)

Show that if $\mathcal{H}$ is a projective plane of order $q$, then $\text{disc}(\mathcal{H}) \geq \sqrt{q}$.

[Pseudo-hint: can be done in essentially one line.]

3. As in class, let

$$f(t) = \max\{|\text{disc}(\mathcal{H})| : \Delta_H = t\}$$

(where $\Delta$ is maximum degree). Show $f(t) \leq 2t - 3$ for $t \geq 3$.

[It’s okay to refer to, rather than repeat, items from class.]

4. Say $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are cross-intersecting if

$$A \cap B \neq \emptyset \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B}.$$  \hspace{1cm} (where $A \cap B \neq \emptyset \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B}$.)

Use Kruskal-Katona to show that if $k + l \leq n \ (k, l \geq 1)$ and $\mathcal{A} \subseteq \binom{n}{k}$, $\mathcal{B} \subseteq \binom{n}{l}$ are cross-intersecting, then either $|\mathcal{A}| \leq \binom{n-1}{k-1}$ or $|\mathcal{B}| < \binom{n-1}{l-1}$.

[Another way (but not here please): shifting as in the proof of EKR.]
5. Show that for $A \subseteq \mathbb{Q}_n$ of size $a$ (and $\log = \log_2$),

$$|\nabla(A)| \geq a(n - \log a).$$

[Your proof should not give (or depend on) the precise result stated in class.]

6. Prove or disprove: For all $k, r \in \mathbb{P}$ there is an $n$ such that for any $f : [n] \times [n] \times [n] \to [r]$ there are $A, B, C \subseteq [n]$ with $|A| = |B| = |C| = k$ and $f$ constant on $A \times B \times C$.

[Note all problem parts have the same weight.]

7. (a) For all $\delta > 0$ and $l \in \mathbb{P}$ there is an $L_0 = L_0(\delta, l)$ for which the following is true. If $L > L_0$, $X$ is a finite set and $Y_1, \ldots, Y_L \subseteq X$ satisfy

$$|Y_i| \geq \delta |X| \quad \forall i \in [L],$$

then there are $a, d \in \mathbb{P}$ such that $(a + (l - 1)d \leq L$ and)

$$Y_a \cap Y_{a+d} \cap \cdots \cap Y_{a+(l-1)d} \neq \emptyset.$$

(b) Show that the statement in (a) implies Szemerédi’s Theorem (in the form: for all $\delta > 0$ and $k \in \mathbb{P}$ there is an $N_0$ such that for any $N > N_0$, any $A \subseteq [N]$ of size at least $\delta N$ contains a $k$-term A.P.).