642.582 Problem Set 3 (first installment)

1. Fix $0 \leq t \leq r \leq n$. [Also, let $M(i,j)$ be the incidence matrix used in class, and assume $R$, $T$ and $U$ range over subsets of $[n]$ of sizes $r$, $t$ and $t-1$ respectively.] Finish Lovász’s proof of EKR by showing that if $M(t-1,t)x = 0$, then for any $R$,

$$\sum_{T \cap R = \emptyset} x_T = (-1)^t \sum_{T \subseteq R} x_T.$$

[Definitions for the next two problems:

For a graph $G = (V, E)$ and $S = (S(v) : v \in V)$ with each $S(v) \subseteq \Gamma (= \{\text{“colors”}\})$, a coloring $\sigma : V \rightarrow \Gamma$ is $S$-legal if it’s proper in the usual sense and $\sigma(v) \in S(v)$ $\forall v \in V$.

The list-chromatic number (or choosability), $\chi_l(G)$, of $G$ is the least $s$ such that every $S$ as above with $|S(v)| = s$ $\forall v$ admits an $S$-legal coloring. Note $\chi_l \geq \chi$ is trivial and the inequality can be strict (e.g. $\chi_l(K_{33}) = 3$).]

2. Show that $\chi_l(G) \leq \lfloor \log_2 n \rfloor + 1 (=: t$, say) for any $n$-vertex bigraph $G$ (say with bipartition $X \cup Y$).

[A-S, 2.7.9]

3. Show that if $\chi(G) = \chi$, then for any $S$ (as above) with $|S(v)| = t$ $\forall v$, there’s an $S$-legal coloring of at least

$$[1 - (1 - 1/\chi)^t]n$$

vertices (where $n = |V(G)|$).

[Easy once found. In the background there’s this lovely problem:

Conjecture. If $\chi_l(G) = s \geq t$, then for any $S$ with $|S(v)| = t$ $\forall v$, there is an $S$-legal coloring of at least $(t/s)n$ vertices.

Note that the $t/s$ can’t be improved in general and—exercise (not to be handed in)—the conjecture is true when $t|s$. As far as I know it’s open in all other cases, e.g. $(s, t) = (3, 2)$.]
4. For a graph $G$, let $G_p$ be the random subgraph gotten by keeping edges independently, each with probability $p$ (e.g. when $G = K_n$, $G_p = G_{n,p}$).

Show that there is a fixed $c > 0$ ($c = 1/2$ will do) for which: if $G = (V, E)$, $|V| = n$ and $\chi(G) = \chi$, then for $H = G_{1/2}$ (and $\log = \log_2$),

$$\mathbb{P}(\chi(H) < c\chi/\log n) = o(1).$$

[Again easy once found, I’m not sure how easy to find. Try to show that $\chi(H) < c\chi/\log n$ implies some other unlikely event. You’re allowed to use:

**Proposition.** For any graph $H$ with chromatic number $\chi$, there is some $W \subseteq V(H)$ with $\delta(H[W]) \geq \chi - 1$ (where $\delta$ is minimum degree).

(The proof is a nice exercise if unfamiliar, but not part of the problem.)]