642.582 Problem Set 3 (final)

1. Fix $0 \le t \le r \le n$. [Also, let $M(i,j)$ be the incidence matrix used in class, and assume R, T and U range over subsets of $[n]$ of sizes r, t and $t-1$ respectively.] Finish Lovász's proof of EKR by showing that if $M(t-1,t)x = \underline{0}$, then for any R,

$$\sum_{T \cap R = \emptyset} x_T = (-1)^t \sum_{T \subseteq R} x_T.$$

[Definitions for the next two problems:

For a graph $G = (V,E)$ and $S = (S(v) : v \in V)$ with each $S(v) \subseteq \Gamma$ ($= \{$"colors"$\}$), a coloring $\sigma : V \to \Gamma$ is *S-legal* if it's proper in the usual sense and $\sigma(v) \in S(v)\ \forall v \in V$.

The *list-chromatic number* (or *choosability*), $\chi_l(G)$, of G is the least s such that every S as above with $|S(v)| = s\ \forall v$ admits an S-legal coloring. Note $\chi_l \ge \chi$ is trivial and the inequality can be strict (e.g. $\chi_l(K_{33}) = 3$).]

2. Show that $\chi_l(G) \le \lfloor \log_2 n \rfloor + 1$ ($=: t$, say) for any n-vertex bigraph G (say with bipartition $X \cup Y$).

[A-S, 2.7.9]

3. Show that if $\chi(G) = \chi$, then for any S (as above) with $|S(v)| = t\ \forall v$, there's an S-legal coloring of at least

$$[1 - (1 - 1/\chi)^t]n$$

vertices (where $n = |V(G)|$).

[Easy *once found*. In the background there's this lovely problem:

Conjecture. If $\chi_l(G) = s \ge t$, then for any S with $|S(v)| = t\ \forall v$, there is an S-legal coloring of at least $(t/s)n$ vertices.

Note that the t/s can't be improved in general and—*exercise* (not to be handed in)—the conjecture is true when $t|s$. As far as I know it's open in all other cases, e.g. $(s,t) = (3,2)$.]

4. For a graph G, let G_p be the random subgraph gotten by keeping edges independently, each with probability p (e.g. when $G = K_n$, $G_p = G_{n,p}$).

Show that there is a fixed $c > 0$ ($c = 1/2$ will do) for which: if $G = (V, E)$, $|V| = n$ and $\chi(G) = \chi$, then for $H = G_{1/2}$ (and $\log = \log_2$),

$$\mathbb{P}(\chi(H) < c\chi/\log n) = o(1).$$

[Again easy once found, I'm not sure how easy to find. Try to show that $\chi(H) < c\chi/\log n$ implies some other unlikely event. You're allowed to use:

Proposition. For any graph H with chromatic number χ, there is some $W \subseteq V(H)$ with $\delta(H[W]) \geq \chi - 1$ (where δ is minimum degree).

(The proof is a nice exercise if unfamiliar, but not part of the problem.)]

5. Show: For each $\varepsilon > 0$ there is an m such that if X is a finite set, $\mathcal{F} \subseteq 2^X$ is nontrivial (i.e. $\mathcal{F} \neq \emptyset, 2^X$) and increasing, and $p = p_c(\mathcal{F})$, then:

$$\text{if } q > mp \text{ then } \mu_q(\mathcal{F}) > 1 - \varepsilon, \text{ and if } q < p/m \text{ then } \mu_q(\mathcal{F}) < \varepsilon.$$

[Recall $p_c(\mathcal{F})$ is defined by $\mu_{p_c(\mathcal{F})}(\mathcal{F}) = 1/2$. This is a sequence-free version of the fact that for any sequence of finite sets $\{X_n\}$ and nontrivial, increasing $\mathcal{F}_n \subseteq 2^{X_n}$, $p_c(\mathcal{F}_n)$ is a threshold in the Erdős-Rényi sense.]

6. An important though easy fact: for any graph G on vertex set V, there's a partition $X \cup Y$ of V such that

$$|\nabla_G(X, Y)| \geq |G|/2$$

(where $\nabla_G(X, Y) = \{e \in G : e \cap X \neq \emptyset \neq e \cap Y\}$; note we regard G as a set of edges).

Proof. If $X \cup Y$ is a uniform partition of V, then $\mathbb{E}|\nabla_G(X, Y)| = |G|/2$. □

[Here's the proof in too much detail, to set possibly useful notation: Let $X \cup Y$ be a uniform partition of V, and for $e \in G$ let $Z_e = \mathbf{1}_{\{e \in \nabla_G(Z,Y)\}}$. Then $Z := \sum_{e \in G} Z_e = |\nabla_G(X, Y)|$, $\mathbb{E}Z_e = 1/2$ and $\mathbb{E}Z = |G|/2$.]

And finally the problem: show that if G, H are *two* graphs on V and $\min\{|G|, |H|\}$ is sufficiently large, then there is a partition $V = X \cup Y$ with

$$|\nabla_G(X, Y)| \geq .49|G| \text{ and } |\nabla_H(X, Y)| \geq .49|H|.$$

7. Let T be the r-branching tree of depth n; thus T has a root, say ρ, at level 0, and for $i \in [n]$, each vertex at level $i-1$ has r children at level i. (We'd usually think of a fixed $r \geq 2$ and large n, but the problem is general.)

Let T_p be the random subtree of T in which each edge of T is present with probability p independent of other choices. (This is *percolation* on T.) Let Q be the event that T_p contains a path from ρ to some leaf of T. Show that for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ (*not* depending on r, n) such that if $p = (1+\varepsilon)/r$ then

$$\mathbb{P}(Q) > \delta.$$

[Please use the following variant of Chebyshev's Inequality (a special case of the *Paley-Zygmund Inequality*).

Proposition. For any *nonnegative* r.v. X, $\mathbb{P}(X > 0) \geq \mu^2/\mathbb{E}X^2$

(where $\mu = \mu_X$). More generally: for *any* X, $\mathbb{P}(X = 0) \leq \sigma_X^2/\mathbb{E}X^2$. The proof is a good small exercise using Cauchy-Schwarz (not to be handed in).

Suggested notation: v, w, x, y for vertices of T; L for the set of leaves of T; P_v for the (unique) (ρ, v)-path in T; $|v|$ for the depth of v in T; $v \wedge w$ for the last vertex in $P_v \cap P_w$ (the most recent common ancestor of v and w); and $v \leq w$ for "v is an *ancestor* of w" (so "$\wedge$" and "$\leq$" apparently think of T growing *upward*).

Solution: don't waste time justifying steps that are clearly okay.]